# Various Characterizations of Optimal Orlicz domains in Sobolev embeddings into Marcinkiewicz spaces

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**Abstract**

We present and follow the method of Optimal Orlicz domains in Sobolev embeddings into Marcinkiewicz spaces [20] for determining whether there exists a largest Orlicz $L^A(\Omega)$ $L^A(\Omega)$ satisfying the Sobolev embedding $W^m L^A(\Omega) \hookrightarrow Y(\Omega)$ $W^m L^A(\Omega) \hookrightarrow Y(\Omega)$

where $Y(\Omega)$$Y(\Omega)$ stands for an arbitrary so-called Marcinkiewicz endpoint space. The tool developed enables us to investigate the optimality of Orlicz domain spaces in Sobolev embeddings and also in Sobolev trace embeddings on domains $\Omega$ $\Omega$in $\mathbb{R}^{(2+\epsilon)}$$\mathbb{R}^{(2+\epsilon)}$ with various regularity.

**Keywords:** Optimal Orlicz domains , Sobolev embedding , Orlicz space , Marcinkiewicz
Spaces

**معايير مختلفة لمجالات أورليش المثلى في طمر سوبوليف داخل فضاءات مارسينكيفيتش**

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**مستخلص**

عرضنا و اتبعنا طريقة مجالات أورليش المثلى في طمر سوبوليف داخل فضاءات مارسينكيفيتش [20] لتحديد ما إذا كان هنالك وجود أوسع لفضاء أورليش $L^A(\Omega)L^A(\Omega)$ يحقق طمر سوبوليف $W^mL^A(\Omega) \hookrightarrow Y(\Omega)W^mL^A(\Omega) \hookrightarrow Y(\Omega)$ عندما يعتمد فضاء باناخ الدالي $Y(\Omega)Y(\Omega)$ على ما يسمى بفضاء مارسينكيفيتش الإختياري للنقطة الحدية . الوسيلة التي طورت تمكننا من فحص الأمثلية في مجال فضاءات أورليش في طمر سوبوليف و أيضا في أثر طمر سوبوليف على مجالات $\Omega\Omega$ في الفضاء الحقيقي $\mathbb{R}^{(2+\epsilon)}\mathbb{R}^{(2+\epsilon)}$ مع إنتظام مختلف.

**كلمات مفتاحية:** مجالات أورليش المثلى- طمر سوبوليف - فضاء أورليش - فضاءات مارسينكيفيتش

## 1. Introduction and main results

For a given Banach function space $Y(\Omega)$,$Y(\Omega)$, we study the question whether there exists an optimal(i.e. largest) Orlicz space $L^A(\Omega)L^A(\Omega)$ satisfying the embedding $W^mL^A(\Omega) \hookrightarrow Y(\Omega)$, $W^mL^A(\Omega) \hookrightarrow Y(\Omega)$, where $\Omega\Omega$ stands for a bounded domain in $\mathbb{R}^{(2+\epsilon)}$, $\epsilon > -2\mathbb{R}^{(2+\epsilon)}$, $\epsilon > -2$and $W^mL^A(\Omega)W^mL^A(\Omega)$ is an Orlicz-Sobolev space. By optimality we mean that the space $L^A(\Omega)L^A(\Omega)$ cannot be replaced by any strictly bigger Orlicz space, i.e., every embedding of an Orlicz-Sobolev space to $Y(\Omega)$ $Y(\Omega)$ factorizes through the space $W^mL^A(\Omega)W^mL^A(\Omega)$.

In general setting of rearrangement-invariant (r.i.) Banach function spaces, such questions were investigated using the method of reducing the Sobolev embeddings to the boundedness of an appropriate modification of the weighted Hardy operator.In the setting of r.i. spaces, the optimal domain and the optimal target spaces are then explicitly described ( see [10,11,16,17]). However, for certain specific applications such as to the solution of partial differential equastions, it is often useful to investigate the optimality of spaces in Sobolev-type embeddings restricted to the context of Orlicz spaces. This creates a difficult and important problem that has been studied by several authors( see e.g. [3-6,8,10,12,13]). In particular, the situation is significantly different than in the broader sense of r.i. spaces.

Consider, for instance, the well-known classical Sobolev embedding

$$W^1L^{1+\epsilon}(\Omega) \hookrightarrow L^{(1+\epsilon)^*}(\Omega), W^1L^{1+\epsilon}(\Omega) \hookrightarrow L^{(1+\epsilon)^*}(\Omega),$$

where $0 < \epsilon < 2\epsilon 0 < \epsilon < 2\epsilon$ $,(1+\epsilon)^* = \left(\frac{2\epsilon^2+3\epsilon+1}{\epsilon}\right)$ $(1+\epsilon)^* = \left(\frac{2\epsilon^2+3\epsilon+1}{\epsilon}\right)$and $\Omega\Omega$ has a Lipschitz boundary.Both the spaces $L^{1+\epsilon}(\Omega)L^{1+\epsilon}(\Omega)$ and $L^{(1+\epsilon)^*}(\Omega)L^{(1+\epsilon)^*}(\Omega)$ that appear in this embedding are clearly optimal in the context of Lebesgue spaces, the former as the domain and the latter as the range. It turns out that they are optimal even in the broader context of Orlicz spaces, but that is a deeper observation and more difficult to prove. The optimality of the range space $L^{(1+\epsilon)^*}(\Omega)L^{(1+\epsilon)^*}(\Omega)$ follows from a general result of A.Cianchi [3].On the other hand, the optimality of the domain space $L^{1+\epsilon}(\Omega)L^{1+\epsilon}(\Omega)$ has not been known so far and will follow from our more general statement below (Example 5.2)).

In the limiting case when $\epsilon = 0\epsilon = 0$, the situation is different and more interesting, First, if we fix the domain space $L^{2+\epsilon}(\Omega)$, $L^{2+\epsilon}(\Omega)$, then there is no optimal range Lebesgue space $L^{\frac{1+\epsilon}{\epsilon}}(\Omega)$ $L^{\frac{1+\epsilon}{\epsilon}}(\Omega)$ that would render the embedding $W^1L^{1+2\epsilon}(\Omega) \hookrightarrow L^{\frac{1+\epsilon}{\epsilon}}(\Omega)$,

$W^1L^{1+2\epsilon}(\Omega) \hookrightarrow L^{\frac{1+\epsilon}{\epsilon}}(\Omega)$, true, because it holds for every $\epsilon < \infty$, $\epsilon < \infty$, but not for $\epsilon = \infty$.$\epsilon = \infty$. This discrepancy was remedied in the 1960s by a clever use of special Orlicz spaces of an exponential type. In particular, by now classic results of N.S. Trudinger,S.I.Pokhozhaev, and V.I.Yudovich (see[25,27]), one has $W^1L^{1+2\epsilon}(\Omega) \hookrightarrow \exp L^{(1+2\epsilon)'}(\Omega)$,$W^1L^{1+2\epsilon}(\Omega) \hookrightarrow \exp L^{(1+2\epsilon)'}(\Omega)$, where $(1+2\epsilon)' = \frac{1+2\epsilon}{2\epsilon}$$(1+2\epsilon)' = \frac{1+2\epsilon}{2\epsilon}$ .Now, both the domain space $L^{1+2\epsilon}(\Omega)$$L^{1+2\epsilon}(\Omega)$ and the range space $\exp L^{(1+2\epsilon)'}(\Omega)$ $\exp L^{(1+2\epsilon)'}(\Omega)$ are Orlicz spaces, and therefore we may ask, again, about their optimality. It turns out that, while the target space is the optimal (that means smallest) Orlicz space that renders this Sobolev embedding true (this was originally proved by J.A.Hempel ,G.R. Morris and N.S.Trudinger in [13]and it also follows from a general result of A.Cianchi [3], the domain space is not.Rather surprisingly, it can even be shown that such an optimal Orlicz domain space does not exist at all.More precisely, given an Orlicz space $L^A(\Omega)$$L^A(\Omega)$ such that $W^1L^A(\Omega) \hookrightarrow \exp L^{(1+2\epsilon)'}(\Omega)$ $W^1L^A(\Omega) \hookrightarrow \exp L^{(1+2\epsilon)'}(\Omega)$, there always exists another Orlicz space $L^B(\Omega)$,$L^B(\Omega)$, strictly bigger than $L^A(\Omega)$$L^A(\Omega)$ such that $W^1L^B(\Omega) \hookrightarrow \exp L^{(1+2\epsilon)'}(\Omega)$.$W^1L^B(\Omega) \hookrightarrow \exp L^{(1+2\epsilon)'}(\Omega)$. This result was shown in [23].

It is clear from the examples that even the very existence of an optimal Orlicz partner (either range or domain) is highly nontrivial and very interesting. However, the question of existence (and, possibly, characterization) of an optimal Orlicz domain partner, is of interest also in a more general situation when the given target space is not necessarily an Orlicz space. For instance, one has the embedding $W^1L^{1+\epsilon}(\Omega) \hookrightarrow L^{\left(\frac{1+\epsilon}{\epsilon}\right)^*,1+\epsilon}(\Omega)$, $W^1L^{1+\epsilon}(\Omega) \hookrightarrow L^{\left(\frac{1+\epsilon}{\epsilon}\right)^*,1+\epsilon}(\Omega)$,(see e.g.[14, 19-22]) in which the target space is a usual two-parameter Lorentz space. Moreover, it is known that $L^{(1+\epsilon)^*,1+\epsilon}(\Omega)L^{(1+\epsilon)^*,1+\epsilon}(\Omega)$ is the optimal r.i. range space in this embedding and the space $L^{1+\epsilon}(\Omega)L^{1+\epsilon}(\Omega)$ is the optimal r.i. domain space(see [11] or [10]). Therefore, $L^{1+\epsilon}(\Omega)$ $L^{1+\epsilon}(\Omega)$ is automatically also the optimal Orlicz space in this embedding.

On the other hand, when we start with the space $L^\infty(\Omega)L^\infty(\Omega)$ at the position of this range space,then, again, as A.Cianchi and L.Pick showed in [7], an optimal Orlicz space does not exist at all. This situation resembles the above-mentioned embedding in which the target was the space $\exp L^{(1+2\epsilon)'}(\Omega).\exp L^{(1+2\epsilon)'}(\Omega)$. Apart from these two very particular cases the question of the existence of an optimal Orlicz space has been open.

The general question of optimality among the Orlicz spaces has already been studied (see [3,4,8-10,12]) however, all those papers focus on the optimality of target spaces.In the case of range ,it turns out the answer is always affirmative, and, furthermore, an explicit description of the optimal Orlicz space is available.The situation is however dramatically different when the target space is fixed and the optimality of the domain space is in question.

Optimal Orlicz domains in Sobolev embeddings into Marcinkiewicz spaces [20] study this question in the special case when the target space is chosen from the class of the so-called Marcinkiewicz endpoint spaces. This is not as restrictive as it may seem since the most customary cases including those given by the previous examples are covered .

An important ingredient of our approach is the use of known reduction theorems (see [10, Theorems 6.1 and 6.4]) and [9, Theoerm 1.3]. This method will enable us to circumvent working with Sobolev spaces to consider instead the boundedness of operator

$$H_{1-\epsilon}^{1+\epsilon}\sum_j f_j(1-\epsilon) := \int_{1+\epsilon}^{1}\sum f_j(1-\epsilon)(1-\epsilon)^{-\epsilon}d(1-\epsilon), \quad 0<\epsilon<1,$$

in one dimension. Here $0<\epsilon<1,0\le\epsilon<\infty$, $0<\epsilon<1,0\le\epsilon<\infty$, and $\epsilon\ge\frac{1}{2}\epsilon\ge\frac{1}{2}$ .Then, by using various special cases of $(1-\epsilon)\,(1-\epsilon)$ and $(1+\epsilon)(1+\epsilon)$we obtain applications not only to Sobolev embeddings but also to the trace Sobolev embeddings of different orders and on various domains in $\mathbb{R}^{1+2\epsilon}\mathbb{R}^{1+2\epsilon}$ at once.

Now we are in a position to state our main result which gives a complete characterization of when the optimal Orlicz domain exists, and also its explicit description. Simply put, to a given Marcinkiewicz endpoint space $M(\Omega)\,M(\Omega)$ we construct an "optimal Orlicz candidate" $L^B(\Omega)L^B(\Omega)$ in terms of the fundamental function. We exploit the fact that to a given fundamental function there always exists a uniquely defined Orlicz space. Next, we test whether the embedding $W^mL^B(\Omega)\hookrightarrow M(\Omega)W^mL^B(\Omega)\hookrightarrow M(\Omega)$ holds. If so, then we show that $L^B(\Omega)L^B(\Omega)$ is the optimal Orlicz domain.Otherwise,we can prove that an optimal Orlicz domain does not exist at all. The general result reads as follows (see [20]).

**Theorem A** Let $0\le\epsilon<1,\epsilon\ge0\;0\le\epsilon<1,\epsilon\ge0$ , $\epsilon\ge\frac{1}{2}\epsilon\ge\frac{1}{2}$ and let $M(0,1)M(0,1)$ be a Marcinkiewicz endpoint space with a fundamental function $\varphi\varphi$

satisfying $\sup_{0<\epsilon<1} \varphi\left((1-\epsilon)^{\frac{1}{1+\epsilon}}\right)(1-\epsilon)^{-\epsilon} = \infty.$

$\sup_{0<\epsilon<1} \varphi\left((1-\epsilon)^{\frac{1}{1+\epsilon}}\right)(1-\epsilon)^{-\epsilon} = \infty.$

Let $X(0,1)X(0,1)$ be the largest r.i. space satisfying $H_{1-\epsilon}^{1+\epsilon} := X(0,1) \to M(0,1)\, H_{1-\epsilon}^{1+\epsilon} := X(0,1) \to M(0,1)$. Denote by $L^B(\Omega)L^B(\Omega)$ the Orlicz space having the same fundamental function as the space $X(0,1).X(0,1)$.Then the following statements are equivalent.

(i) There exists a largest Orlicz space $L^A(\Omega)L^A(\Omega)$ satisfying the relation $H_{1-\epsilon}^{1+\epsilon} : L^A(0,1) \to M(0,1); H_{1-\epsilon}^{1+\epsilon} : L^A(0,1) \to M(0,1);$

(ii) $H_{1-\epsilon}^{1+\epsilon} : L^B(0,1) \to M(0,1); H_{1-\epsilon}^{1+\epsilon} : L^B(0,1) \to M(0,1);$

(iii) $L^B(0,1) \subseteq X(0,1); L^B(0,1) \subseteq X(0,1);$

(iv) $(1-\epsilon)_{1-\epsilon} : L^{\tilde{B}}(0,1) \to L^{\tilde{B}}(0,1),$ $(1-\epsilon)_{1-\epsilon} : L^{\tilde{B}}(0,1) \to L^{\tilde{B}}(0,1),$ where $(1-\epsilon)_{1-\epsilon}(1-\epsilon)_{1-\epsilon}$ is the operator given by

$$\sum\left((1-\epsilon)_{1-\epsilon} f_j\right)(1-\epsilon) := (1-\varepsilon)^{-\epsilon} \sup_{0<\epsilon<\frac{1}{2}} \sum (1-\epsilon)^{-\epsilon} f_j^*(1-\epsilon),$$

$$\sum\left((1-\epsilon)_{1-\epsilon} f_j\right)(1-\epsilon) := (1-\varepsilon)^{-\epsilon} \sup_{0<\epsilon<\frac{1}{2}} \sum (1-\epsilon)^{-\epsilon} f_j^*(1-\epsilon),$$

$0 < \epsilon < 1;$

(v) there exists some $\epsilon \geq 0 \epsilon \geq 0$ such that

$$\int_1^{2+\epsilon} \frac{\tilde{B}(1-\epsilon)}{(1-\epsilon)^{\frac{-(1+\epsilon)}{\epsilon}}}\, d(1-\epsilon) \lesssim \frac{\tilde{B}(\epsilon^2+3\epsilon+2)}{(2+\epsilon)^{\frac{-1}{\epsilon}}},$$

$$\int_1^{2+\epsilon} \frac{\tilde{B}(1-\epsilon)}{(1-\epsilon)^{\frac{-(1+\epsilon)}{\epsilon}}}\, d(1-\epsilon) \lesssim \frac{\tilde{B}(\epsilon^2+3\epsilon+2)}{(2+\epsilon)^{\frac{-1}{\epsilon}}},$$

$0 < \epsilon < \infty,. 0 < \epsilon < \infty,.$ Moreover, if $\tilde{B}\tilde{B}$ satisfies the $\Delta_2\Delta_2$ condition, then each of the conditions(i)-(v) is equivalent to the following statement:

(vi) there exists some $\epsilon \geq 0\epsilon \geq 0$ such that $\lim_{1-\epsilon\to\infty} \sup \frac{\tilde{B}(2+\epsilon)}{\tilde{B}(\epsilon^2+3\epsilon+2)} < (1+\epsilon)^{\frac{-1}{\epsilon}}$ $\lim_{1-\epsilon\to\infty} \sup \frac{\tilde{B}(2+\epsilon)}{\tilde{B}(\epsilon^2+3\epsilon+2)} < (1+\epsilon)^{\frac{-1}{\epsilon}}$

Note that the condition on $\varphi\varphi$ causes no loss of generality, since otherwise

$H_{1-\epsilon}^{1+\epsilon}: L^1(0,1) \to M(0,1). H_{1-\epsilon}^{1+\epsilon}: L^1(0,1) \to M(0,1)$. The details are discussed in Remark 3.7.

The proof of Theorem A relies on the next result of independent interest, which provides us with a reduction theorem for Orlicz and Marcinkiewicz spaces (see [20]).

**Theorem B** Let $0 \leq \epsilon < 1, \epsilon \geq \frac{1}{2} 0 \leq \epsilon < 1, \epsilon \geq \frac{1}{2}$ and let $L^A(0,1)L^A(0,1)$ be an Orlicz space with a Young function $A$ $A$ and $M(0,1)M(0,1)$be a Marcinkiewicz endpoint space with a fundamental function $\varphi$ $\varphi$satisfying

$$\sup_{2>\epsilon>1} \varphi\left((2+\epsilon)^{\frac{1}{1+\epsilon}}\right)(1-\epsilon)^{-\epsilon} = \infty.$$

$\sup_{2>\epsilon>1} \varphi\left((2+\epsilon)^{\frac{1}{1+\epsilon}}\right)(1-\epsilon)^{-\epsilon} = \infty.$ Then the relation $H_{1-\epsilon}^{1+\epsilon}: L^A(0,1) \to M(0,1)H_{1-\epsilon}^{1+\epsilon}: L^A(0,1) \to M(0,1)$ holds if and only if there exists $C > 0C > 0$ such that

$$\int_1^{2+\epsilon} \frac{\tilde{A}(1-\epsilon)}{(1-\epsilon)^{\frac{-(1+\epsilon)}{\epsilon}}} d(1-\epsilon) \lesssim \frac{\tilde{B}\left(C(2+\epsilon)\right)}{(2+\epsilon)^{\frac{-1}{\epsilon}}}, \quad 0 < \epsilon < \infty,$$

where $B$ $B$ is a Young function described in Theorem A and $\tilde{A}$ $\tilde{A}$and $\tilde{B}\tilde{B}$ are complementary Young functions to $AA$ and $BB$ respectively.

Our final principal result describes the fundamental function of the optimal r.i. domain space (see [20]).

**Theorem C** $0 \le \epsilon < 1, \epsilon \ge \frac{1}{2}$. $0 \le \epsilon < 1, \epsilon \ge \frac{1}{2}$. Suppose that $M(0,1) M(0,1)$ is the Marcinkiewicz endpoint space with a fundamental function $\varphi$ $\varphi$. Then the fundamental function $\varphi_X$ $\varphi_X$ of the largest r.i. space $X(0,1) X(0,1)$ having the property $H_{1-\epsilon}^{1+\epsilon}: X(0,1) \to M(0,1)$ $H_{1-\epsilon}^{1+\epsilon}: X(0,1) \to M(0,1)$ satisfies

$$\varphi_X(1-\epsilon) \simeq (1-\epsilon) \sup_{0<\epsilon<1} \varphi\left((1-\epsilon)^{\frac{1}{1+\epsilon}}\right)(1-\epsilon)^{-\epsilon}, \quad 0 < \epsilon < 1.$$

The paper is structured as follows. In Section 2 we collect all the necessary basic background material. In Section 3 we prove Theorem B and Theorem C. In Section 4 we prove Theorem A. Finally, Section 5 contains various applications and examples of the main result.

## 2. Function spaces

By $A \lesssim B A \lesssim B$ and $A \gtrsim B A \gtrsim B$ we mean that $A \le CB A \le CB$ and $A \ge CB A \ge CB$, respectively, where $C C$ is a positive constant independent of the appropriate quantites involved in $A$ $A$ and $B$ $B$. We shall write $A \simeq B A \simeq B$ when both of the estimates $A \lesssim B$ $A \lesssim B$ and $A \gtrsim B A \gtrsim B$ are satisfied. We shall use the convention

$0.\infty = 0, \frac{0}{0} = 0 0.\infty = 0, \frac{0}{0} = 0$ and $\frac{\infty}{\infty} = 0 \frac{\infty}{\infty} = 0$.

When $X X$ and $Y Y$ are Banach spaces, we say that $X X$ is embedded into $Y Y$, and write $X \hookrightarrow Y X \hookrightarrow Y$, if $X \subseteq Y X \subseteq Y$ and there exists a positive constant $C C$, such that $\left\|\sum_j f_j\right\|_Y \le C \sum \left\|f_j\right\|_X$ $\left\|\sum_j f_j\right\|_Y \le C \sum \left\|f_j\right\|_X$ for every $f_j \in X . f_j \in X$. We say that a linear operator $T T$ defined on $X X$ with values in $Y Y$ is bounded if there exists a constant $C > 0C > 0$ such that $\|\sum_j f_j\|_Y \leq C\sum\|f_j\|_X$ $\|\sum_j f_j\|_Y \leq C\sum\|f_j\|_X$ for every $f_j \in X f_j \in X$. We write $T: X \to Y$ $T: X \to Y$ in this case.

We say that a function $G: [0,\infty) \to (0,\infty) G: [0,\infty) \to (0,\infty)$ satisfies the $\Delta_2 \Delta_2$ condition at infinity if there exist $\epsilon > -1$ $\epsilon > -1$ and $T \geq 0 T \geq 0$ such that $G(4+2\epsilon) \leq (1+\epsilon)G(2+\epsilon)$ $G(4+2\epsilon) \leq (1+\epsilon)G(2+\epsilon)$ for every $2+\epsilon \geq T. 2+\epsilon \geq T$. We will use only $\Delta_2 \Delta_2$ condition at infinity, hence we shall shortly say $\Delta_2 \Delta_2$ condition and write $G \in \Delta_2 . G \in \Delta_2$.

For a nonnegative functions $f_j f_j$ we shall write $\int_0 \sum f_j < \infty$ $\int_0 \sum f_j < \infty$ when there exists some $c > 0 c > 0$ such that the integral $\int_0^c \sum f_j \int_0^c \sum f_j$ converges. By integral we always mean the Lebesgue integral.

### 2.1 Rearrangement-invariant spaces

We recall definitions and some basic facts concerning the rearrangement-invariant spaces, which we will need in the following text. We shall not prove well-known results, all proofs and futher details can be found in C. Bennett and R.Sharpley [1].

Suppose $\Omega \Omega$ is a domain in $\mathbb{R}^{(1+2\epsilon)} . \mathbb{R}^{(1+2\epsilon)}$. Let $\mathcal{M}(\Omega) \mathcal{M}(\Omega)$ be a class of real-valued measurable functions on $\Omega \Omega$ and $\mathcal{M}^+(\Omega)$ $\mathcal{M}^+(\Omega)$ the class of nonnegative functions in $\mathcal{M}(\Omega) . \mathcal{M}(\Omega)$. Given $f_j \in \mathcal{M} f_j \in \mathcal{M}$ we define its nonincreasing rearrangement on $(0,|\Omega|)(0,|\Omega|)$ as

$$\sum f_j^*(1+\epsilon) := \inf\left\{\lambda > 0, \mu_{\sum f_j}(\lambda) \leq 1+\epsilon\right\}, \qquad 0 \leq \epsilon < |\Omega|,$$

where $\mu_{\sum f_i}\mu_{\sum f_i}$ is the distribution function of $f_j f_j$, i.e.,

$$\mu_{\sum f_j}(\lambda) := \left|\left\{x \in \Omega, \sum |f_j(x)| > \lambda\right\}\right|, \lambda > 0,$$

where the $|\cdot|$ $|\cdot|$ stands for the Lebesgue measure. The Hardy average $f_j^{**}$ $f_j^{**}$ is defined on $(0,|\Omega|)(0,|\Omega|)$ as

$$\sum f_j^{**}(1+\epsilon) = \frac{1}{1+\epsilon}\int_0^{1+\epsilon} \sum f_j^{*}(1-\epsilon)d(1-\epsilon), \ 0 \le \epsilon < |\Omega|.$$

$$\sum f_j^{**}(1+\epsilon) = \frac{1}{1+\epsilon}\int_0^{1+\epsilon} \sum f_j^{*}(1-\epsilon)d(1-\epsilon), \ 0 \le \epsilon < |\Omega|.$$

Let $f_j, g_j \in \mathcal{M}^+(\Omega).f_j, g_j \in \mathcal{M}^+(\Omega)$. Then we have the Hardy-Littlewood inequality

$$\int_\Omega \sum f_j(x)g_j(x)dx \le \int_0^{|\Omega|} \sum f_j^*(1+\epsilon)g_j^*(1+\epsilon)d(1+\epsilon).$$

When $E \subseteq \Omega E \subseteq \Omega$ is measurable, we denote by $\chi_E \chi_E$ the characteristic function of $EE$. A simple function is afinite sum $\sum_j \lambda_j \chi_{E_j,} \sum_j \lambda_j \chi_{E_j,}$ where $\lambda_j \neq 0 \lambda_j \neq 0$ is a real number and $E_j \subseteq \Omega E_j \subseteq \Omega$ has finite measure for every index $jj$.

Denote by $II$ the interval $(0,1).(0,1)$. A mapping $\rho: \mathcal{M}^+(I) \to [0,\infty]$ $\rho: \mathcal{M}^+(I) \to [0,\infty]$ is called a rearrangement-invariant (r.i.) Banach function norm on $\mathcal{M}^+(I)\mathcal{M}^+(I)$ if for all

$f_j, g_j, f_{(1+2\epsilon)}(1+2\epsilon \in \mathbb{N})$ in $\mathcal{M}^+(I)$, $f_j, g_j, f_{(1+2\epsilon)}(1+2\epsilon \in \mathbb{N})$ in $\mathcal{M}^+(I)$, for all constants $a \ge 0$ $a \ge 0$ and for every measurable subset $E\,E$ of $I\,I$, the following properties hold:

(P1) $\rho(f_j) = 0 \leftrightarrow f_j = 0 \rho(f_j) = 0 \leftrightarrow f_j = 0$ a.e.; $\rho(af_j) = a\rho(f_j)$; $\rho(f_j + g_j) \le \rho(f_j) + \rho(g_j)$;

$\rho(af_j) = a\rho(f_j)$; $\rho(f_j + g_j) \leq \rho(f_j) + \rho(g_j)$;

(P2) $0 \leq f_j \leq g_j 0 \leq f_j \leq g_j$ a.e. implies $\rho(f_j) \leq \rho(g_j)$; $\rho(f_j) \leq \rho(g_j)$;

(P3) $0 \leq f_{(1+2\epsilon)} \uparrow f_j 0 \leq f_{(1+2\epsilon)} \uparrow f_j$ a.e. implies $\rho(f_{(1+2\epsilon)}) \uparrow \rho(f_j)$;$\rho(f_{(1+2\epsilon)}) \uparrow \rho(f_j)$;

(P4) $\rho(\chi_I) < \infty$;$\rho(\chi_I) < \infty$;

(P5) $\int_0^1 \sum f_j(x)\, dx \lesssim \rho(f_j)$;$\int_0^1 \sum f_j(x)\, dx \lesssim \rho(f_j)$;

(P6) $\rho(f_j) = \rho(f_j^*)$.$\rho(f_j) = \rho(f_j^*)$.

The associate norm of an r.i. norm $\rho\rho$ is another such norm $\rho'\rho'$ defined as

$$\rho'(g_j) := \sup_{\rho(f_j) \leq 1} \int_0^1 \sum g_j(1+\epsilon) f_j(1+\epsilon)\, d(1+\epsilon), f_j, g_j \in \mathcal{M}^+(I).$$

It obeys the principle of Duality; that is $\rho'' := (\rho')' = \rho$. $\rho'' := (\rho')' = \rho$.

Furthermore, the Hölder inequality

$\int_0^1 \sum f_j(1+\epsilon) g_j(1+\epsilon)\, d(1+\epsilon)\, \rho(f_j)\rho'(g_j)$

$\int_0^1 \sum f_j(1+\epsilon) g_j(1+\epsilon)\, d(1+\epsilon)\, \rho(f_j)\rho'(g_j)$holds for every $f_j, g_j \in \mathcal{M}^+(I)$.$f_j, g_j \in \mathcal{M}^+(I)$.

Given the r.i. norm $\rho$,$\rho$, the corresponding rearrangement-invariant Banach function space or, for short, r.i. space, is the collection

$L_\rho(I) := \{f_j \in \mathcal{M}(I), \rho(|f_j|) < \infty\}$,

$L_\rho(I) := \{f_i \in \mathcal{M}(I), \rho(|f_i|) < \infty\}$,endowed with the norm $\sum \|f_j\|_{L_\rho(I)} := \rho(|f_j|), \quad f_j \in L_\rho(I)$.

Next, given a bounded domain $\Omega\Omega$ and $\mathbb{R}^{(1+2\epsilon)},\mathbb{R}^{(1+2\epsilon)}$, we define the r.i. space

$$L_\rho(\Omega) := \left\{f_j \in \mathcal{M}(\Omega), \rho\left(\sum f_j^*((1+\epsilon)|\Omega|)\right) < \infty\right\}$$

with $\sum\|f_j\|_{L_\rho(\Omega)} := \rho(\sum f_j^*((1+\epsilon)|\Omega|)), \quad f_j \in L_\rho(\Omega).$

$\sum\|f_j\|_{L_\rho(\Omega)} := \rho(\sum f_j^*((1+\epsilon)|\Omega|)), \quad f_j \in L_\rho(\Omega).$

If $\rho_1\rho_1$ and $\rho_2\rho_2$ are two r.i. norms, then $L_{\rho_1}(\Omega) \subseteq L_{\rho_2}(\Omega)$ $L_{\rho_1}(\Omega) \subseteq L_{\rho_2}(\Omega)$implies $L_{\rho_1}(\Omega) \hookrightarrow L_{\rho_2}(\Omega)L_{\rho_1}(\Omega) \hookrightarrow L_{\rho_2}(\Omega)$. Let $\varphi\ \varphi$ be nonnegative function defined on the interval $[0,\infty)\ [0,\infty)$ . If

(i) $\varphi(1+\epsilon) = 0\ \varphi(1+\epsilon) = 0$ iff $\epsilon = -1, \epsilon = -1$,

(ii) $\varphi(1+\epsilon)\varphi(1+\epsilon)$ is nondecreasing on $(0,\infty)(0,\infty)$,

(iii) $\frac{\varphi(1+\epsilon)\varphi(1+\epsilon)}{1+\epsilon \quad 1+\epsilon}$ is nonincreasing on $(0,\infty)(0,\infty)$,

then $\varphi\ \varphi$ is said to be quasiconcave. We also say that a function $\varphi\varphi$ defined on bounded interval $[0,R],[0,R]$, for $R \in (0,\infty)$ $R \in (0,\infty)$, is quasiconcave if the continuation by constant value $\varphi(R)\varphi(R)$ is quasiconcave on $[0,\infty)[0,\infty)$.

The fundamental function of an r.i. norm $\rho\ \rho$ on $\mathcal{M}^+(I)\mathcal{M}^+(I)$ is defined by

$$\varphi_\rho(1+\epsilon) := \rho(\chi_{(0,1+\epsilon)}), \qquad 1+\epsilon \in I, \varphi_\rho(0) = 0.$$

The fundamental function is quasiconcave on $[0,1)$ $[0,1)$, continuous except perhaps at the origin and satisfies $\varphi_\rho(1+\epsilon)\varphi_{\rho'}(1+\epsilon) = 1+\epsilon,\ 1+\epsilon \in I.$ $\varphi_\rho(1+\epsilon)\varphi_{\rho'}(1+\epsilon) = 1+\epsilon,\ 1+\epsilon \in I.$

Quasiconcave functions need not be concave, however, every r.i. space can be equivalently renormed so that its fundamental function is concave. Let $\varphi\varphi$ be a concave function. We define the Lorentz endpoint space $\Lambda_\varphi(\Omega)\,\Lambda_\varphi(\Omega)$ by the function norm

$$\rho_{\Lambda_\varphi}(f_j) := \int_0^1 \sum f_j^*(1+\epsilon)d_\varphi(1+\epsilon), \quad f_j \in \mathcal{M}^+(I),$$

where $d_\varphi\, d_\varphi$ stands for the Lebesgue-Stieltjes measure associated with $\varphi.\varphi$. We define the Marcinkiewicz endpoint space $M_\varphi(\Omega)$ $M_\varphi(\Omega)$ by the function norm

$$\rho_{M_\varphi}(f_j) := \sup_{-1<\epsilon<0} f_j^{**}(1+\epsilon)\varphi(1+\epsilon), \quad f_j \in \mathcal{M}^+(I).$$

The endpoint spaces $\Lambda_\varphi(\Omega)\Lambda_\varphi(\Omega)$ and $M_\varphi(\Omega)M_\varphi(\Omega)$ are r.i. spaces with the fundamental function $\varphi.\varphi$. If $X(\Omega)\,X(\Omega)$ is an r.i. space with the fundamental function $\varphi,\varphi$, then $\Lambda_\varphi(\Omega) \hookrightarrow X \hookrightarrow M_\varphi(\Omega)$. $\Lambda_\varphi(\Omega) \hookrightarrow X \hookrightarrow M_\varphi(\Omega)$.

In other words, $\Lambda_\varphi(\Omega)\Lambda_\varphi(\Omega)$ and $M_\varphi(\Omega)M_\varphi(\Omega)$ are respectively the smallest and the largest r.i. spaces having the fundamental function equivalent to $\varphi.\,\varphi$. The associate space of a Lorentz endpoint space $\Lambda_\varphi\,\Lambda_\varphi$ is the Marcinkiewicz endpoint space $M_\psi M_\psi$ where both $\varphi\varphi$ and $\psi\psi$ are concave and $\varphi(1+\epsilon)\psi(1+\epsilon) = 1+\epsilon$ $\varphi(1+\epsilon)\psi(1+\epsilon) = 1+\epsilon$ on $I\,I$. If $|\Omega| < \infty, |\Omega| < \infty$, then for every r.i. space $X(\Omega)L^\infty(\Omega) \hookrightarrow X(\Omega) \hookrightarrow L^1(\Omega)$. $X(\Omega)L^\infty(\Omega) \hookrightarrow X(\Omega) \hookrightarrow L^1(\Omega)$.

Assume either $0 \le \epsilon \le \infty 0 \le \epsilon \le \infty$. The Lorentz space $L^{(1+\epsilon,1+2\epsilon)}(\Omega)L^{(1+\epsilon,1+2\epsilon)}(\Omega)$ is defined by the functional

$$\rho_{(1+\epsilon,1+2\epsilon)}(f_j) = \rho_{(1+2\epsilon)}\left((1+\epsilon)^{\frac{\epsilon}{2\epsilon^2+3\epsilon+1}} f_j^*(1+\epsilon)\right), f_j \in \mathcal{M}^+(I),$$

$$\rho_{(1+\epsilon,1+2\epsilon)}(f_j) = \rho_{(1+2\epsilon)}\left((1+\epsilon)^{\frac{\epsilon}{2\epsilon^2+3\epsilon+1}} f_j^*(1+\epsilon)\right), f_j \in \mathcal{M}^+(I),$$

where

$$\rho_{(1+2\epsilon)}(f_j) = \begin{cases} \left(\int_0^1 \sum f_j (1+\epsilon)^{1+2\epsilon} d(1+\epsilon)\right)^{\frac{1}{1+2\epsilon}}, & 0 \le \epsilon < \infty, \\ \operatorname*{ess\ sup}_{-1<\epsilon<0} f_j(1+\epsilon), & \epsilon = 0, \end{cases}$$

Stands for the Banach function norm of the Lebesgue space $L^{1+\epsilon}(\Omega).L^{1+\epsilon}(\Omega)$. The functional $\rho_{(1+2\epsilon,1+\epsilon)}\rho_{(1+2\epsilon,1+\epsilon)}$ is a Banach function norm if and only if $0 \le \epsilon \le 10 \le \epsilon \le 1$. However, for $0 < \epsilon < \infty, \rho_{(1+2\epsilon,1+\epsilon)} 0 < \epsilon < \infty, \rho_{(1+2\epsilon,1+\epsilon)}$ can be equivalently replaced by Banach function norm

$$\rho_{(1+2\epsilon,1+\epsilon)}(f_j) = \rho_{(1+\epsilon)}\left((1+\epsilon)^{\frac{\epsilon}{2\epsilon^2+3\epsilon+1}} f_j^{**}(1+\epsilon)\right).$$

$$\rho_{(1+2\epsilon,1+\epsilon)}(f_j) = \rho_{(1+\epsilon)}\left((1+\epsilon)^{\frac{\epsilon}{2\epsilon^2+3\epsilon+1}} f_j^{**}(1+\epsilon)\right).$$

The fundamental function of the norm $\rho_{(1+2\epsilon,1+\epsilon)}\ \rho_{(1+2\epsilon,1+\epsilon)}$ satisfies

$$\varphi_{\rho_{(1+2\epsilon,1+\epsilon)}}(1+\epsilon) \simeq (1+\epsilon)^{\frac{1}{1+2\epsilon}}, \qquad 0 < \epsilon < 1.$$

$$\varphi_{\rho_{(1+2\epsilon,1+\epsilon)}}(1+\epsilon) \simeq (1+\epsilon)^{\frac{1}{1+2\epsilon}}, \qquad 0 < \epsilon < 1.$$

The spaces $L^{(1+2\epsilon,1)}(\Omega)L^{(1+2\epsilon,1)}(\Omega)$ and $L^{(1+2\epsilon,\infty)}(\Omega)L^{(1+2\epsilon,\infty)}(\Omega)$ are equal to the Lorentz and Marcinkiewicz endpoint spaces $\Lambda_\varphi(\Omega)\Lambda_\varphi(\Omega)$ and $M_\varphi(\Omega)M_\varphi(\Omega)$, respectively, with

$\varphi(1+\epsilon)=(1+\epsilon)^{\frac{1}{1+2\epsilon}}\varphi(1+\epsilon)=(1+\epsilon)^{\frac{1}{1+2\epsilon}}$. If the first parameter is fixed, then the Lorentz spaces are nested, i.e., we have $L^{(1+\epsilon,1+\epsilon)}(\Omega)\hookrightarrow L^{(1+\epsilon,1+2\epsilon)}(\Omega)L^{(1+\epsilon,1+\epsilon)}(\Omega)\hookrightarrow L^{(1+\epsilon,1+2\epsilon)}(\Omega)$

whenever $0\le\epsilon\le\infty, 0\le\epsilon\le\infty,$

**2.2.Orlicz spaces**

We also need to know defenitions and all the basic facts about Young functions and Orlicz Spaces. All of these can be found for instanc in the book by L. Pick, A. Kufner, O. John and S. Fučĭk [24].We shall say that A is a Young function if there exists a function $a:[0,\infty)\to[0,\infty)a:[0,\infty)\to[0,\infty)$ such that $A(1+\epsilon)=\int_0^{1+\epsilon}a(1-\epsilon)d(1-\epsilon),\quad -1\le\epsilon<\infty,$ $A(1+\epsilon)=\int_0^{1+\epsilon}a(1-\epsilon)d(1-\epsilon),\quad -1\le\epsilon<\infty,$

and $a$ $a$ has the following properties:

(i) $a(1-\epsilon)>0a(1-\epsilon)>0$ for $\epsilon>1,\ a(0)=0;$ $\epsilon>1,\ a(0)=0;$

(ii) $aa$ is right-continuous;

(iii) $aa$ is nondecreasing;

(iv) $\lim_{1-\epsilon\to\infty}a(1-\epsilon)=\infty.\lim_{1-\epsilon\to\infty}a(1-\epsilon)=\infty.$

Every Young function is continuous, nonnegative, strictly increasing, convex on $[0,\infty)[0,\infty)$ and satisfies $\lim_{1+\epsilon\to0^+}\frac{A(1+\epsilon)}{1+\epsilon}=\lim_{1+\epsilon\to\infty}\frac{1+\epsilon}{A(1+\epsilon)}=0.$ $\lim_{1+\epsilon\to0^+}\frac{A(1+\epsilon)}{1+\epsilon}=\lim_{1+\epsilon\to\infty}\frac{1+\epsilon}{A(1+\epsilon)}=0.$

Furthermore, one has

$A(\epsilon^2+2\epsilon+1)\le(1+\epsilon)A(1+\epsilon),\ 0\ge\epsilon\ge-1,$ $A(\epsilon^2+2\epsilon+1)\le(1+\epsilon)A(1+\epsilon),\ 0\ge\epsilon\ge-1,$ and

$A(\epsilon^2+2\epsilon+1) \geq (1+\epsilon)A(1+\epsilon), 0<\epsilon<\infty, \qquad \epsilon \geq -1.$

Moreover $\frac{A(1+\epsilon)}{1+\epsilon}\frac{A(1+\epsilon)}{1+\epsilon}$ is increasing on $(1,\infty)(1,\infty)$ and we have the estimates
$A(1+\epsilon) \leq a(1+\epsilon)(1+\epsilon) \leq 2A(1+\epsilon), -1<\epsilon<\infty$ .

For a Young function $AA$ and a domain $\Omega \subseteq \mathbb{R}^{(1+2\epsilon)}, \Omega \subseteq \mathbb{R}^{(1+2\epsilon)}$, the Orlicz space $L^A = L^A(\Omega) L^A = L^A(\Omega)$ is the collection of all functions $f_j \in \mathcal{M}(\Omega) f_j \in \mathcal{M}(\Omega)$ for which there exists a $\lambda > 0$ $\lambda > 0$ such that $\int_\Omega A\left(\sum \frac{|f_j(x)|}{\lambda}\right) dx < \infty . \int_\Omega A\left(\sum \frac{|f_j(x)|}{\lambda}\right) dx < \infty.$
The Orlicz space $L^A(\Omega)\, L^A(\Omega)$ is endowed with the Luxemburg norm

$$\left\|\sum_j f_j\right\|_{L^A} := \inf\left\{\lambda > 0, \int_\Omega A\left(\sum_j \frac{|f_j(x)|}{\lambda}\right) dx \leq 1\right\}.$$

The complementary function $\tilde{A}\tilde{A}$ is a Young function $AA$ is given by

$\tilde{A}(1+\epsilon) := \sup_{\epsilon<1}\left((1-\epsilon^2) - A(1-\epsilon)\right), \ -1 \leq \epsilon < \infty,$

$\tilde{A}(1+\epsilon) := \sup_{\epsilon<1}\left((1-\epsilon^2) - A(1-\epsilon)\right), \ -1 \leq \epsilon < \infty,.$

The complementary function $\tilde{A}\tilde{A}$ is a Young function as well and the complementary function of $\tilde{A}\tilde{A}$ is once more $AA$. For any Young function $AA$ and its complementary function $\tilde{A}\tilde{A}$ there is the relation

$1+\epsilon \leq A^{-1}(1+\epsilon)\tilde{A}^{-1}(1+\epsilon) \leq 2(1+\epsilon), -1 \leq \epsilon < \infty$

$1+\epsilon \leq A^{-1}(1+\epsilon)\tilde{A}^{-1}(1+\epsilon) \leq 2(1+\epsilon), -1 \leq \epsilon < \infty.$

With the help of the complementary function we can define an alternative Orlicz norm on an Orlicz space by

$$\left\|\sum_j f_j\right\|_{L^A} := \sup\left\{\int_\Omega \sum_j |f_j(x)g_j(x)|dx\right\},$$

where the supremum is taken over all functions $g_j \in \mathcal{M}(\Omega)$ $g_j \in \mathcal{M}(\Omega)$ such that

$\int_\Omega \sum \tilde{A}\left(g_j(x)\right)dx \le 1.$ $\int_\Omega \sum \tilde{A}\left(g_j(x)\right)dx \le 1.$ The Luxemburg and Orlicz norms are equivalent, namely,

$$\left\|\sum_j f_j\right\|_{L^A} \le \sum_j \|f_j\|_{(L^A)} \le 2\sum_j \|f_j\|_{L^A}.$$

When $L^A(\Omega)L^A(\Omega)$ is an Orlicz space endowed with the Luxemburg norm, then the associate space $L^{\tilde{A}}(\Omega)L^{\tilde{A}}(\Omega)$ with the Orlicz norm. In particular, the sharp Hölder inequality for Orlicz spaces has the form $\int_\Omega \sum|f_j(x)g_j(x)|\,dx \le \sum\|f_j\|_{L^A}\sum\|f_j\|_{(L^{\tilde{A}})}$

$\int_\Omega \sum|f_j(x)g_j(x)|\,dx \le \sum\|f_j\|_{L^A}\sum\|f_j\|_{(L^{\tilde{A}})}$

The Orlicz spaces $L^A(\Omega)L^A(\Omega)$ is an r.i. space and $\|\chi_E\|_{L^A} = \frac{1}{A^{-1}\left(\frac{1}{|E|}\right)}$ $\|\chi_E\|_{L^A} = \frac{1}{A^{-1}\left(\frac{1}{|E|}\right)}$ for every measurable $E \subseteq \Omega E \subseteq \Omega$ of positive measure, thus, for a bounded domain $\Omega,\Omega$, the fundamental function for the Luxemburg norm is

$$\varphi_{L^A}(1+\epsilon) = \frac{1}{A^{-1}\left(\frac{1}{|\Omega|}\right)}, \qquad 1+\epsilon \in I, \quad \varphi_{L^A}(0) = 0.$$

An Orlicz space $L^A(I)L^A(I)$ with fundamental function $\varphi\varphi$

coincides with the Marcinkiewicz endpoint space $M_\varphi(I)M_\varphi(I)$ if and only if there exists $\delta \in (0,1)\delta \in (0,1)$ such that

$$\int_0^1 A\left(\delta A^{-1}\left(\frac{1}{1+\epsilon}\right)\right) d(1+\epsilon) < \infty \quad (2.1)$$

(see also [18]).

For $|\Omega| < \infty, |\Omega| < \infty$, the inclusion relation between Orlicz spaces is governed by inequalities involving the corresponding Young functions.If $AA$ and $BB$ are Young functions then $L^A(\Omega) \hookrightarrow L^B(\Omega) L^A(\Omega) \hookrightarrow L^B(\Omega)$ if and only if there exist $c > 0$ $c > 0$ and $T \geq 0 T \geq 0$ such that

$$B(1+\epsilon) \leq A(c(1+\epsilon)),\ 1+\epsilon \geq T,$$

$B(1+\epsilon) \leq A(c(1+\epsilon)),\ 1+\epsilon \geq T,$ which we denote by $B \prec A$ $B \prec A$or $A \succ BA \succ B$. If both $A \prec BA \prec B$ and $A \succ BA \succ B$ hold, we say that$AA$and $BB$ are equivalent and write $A \approx B$. $A \approx B$. When $|\Omega| < \infty, |\Omega| < \infty$, the inclusion $L^A(\Omega) \subseteq L^B(\Omega)$ $L^A(\Omega) \subseteq L^B(\Omega)$ is proper if and only if $\lim_{1+\epsilon\to\infty} \sup \frac{B(1+\epsilon)}{A(\lambda(1+\epsilon))} = 0$

$\lim_{1+\epsilon\to\infty} \sup \frac{B(1+\epsilon)}{A(\lambda(1+\epsilon))} = 0$

for every $\lambda > 0$. $\lambda > 0$. In such case we write $B \prec\prec AB \prec\prec A$ or $A \succ\succ B.A \succ\succ B$. If $A \prec BA \prec B$ or $A \prec\prec BA \prec\prec B$ then $\tilde{A} \succ \tilde{B}\tilde{A} \succ \tilde{B}$ or $\tilde{A} \succ\succ \tilde{B},\tilde{A} \succ\succ \tilde{B}$, respectively.

## 3.Proofs of Theorems B and C.

**Lemma 3.1.** Let $AA$ be a Young function and let $\xi\xi$ be a nonzero real number. Assuming

$$\int_0 A(1-\epsilon)(1-\epsilon)^{\frac{1}{\xi}-1} d(1-\epsilon) < \infty, \qquad (3.1)$$

we define

$$E_\xi(1+\epsilon) = |\xi|^{-1}(1+\epsilon)^{-\frac{1}{\xi}} \int_0 A(1-\epsilon)(1-\epsilon)^{\frac{1}{\xi}-1} d(1-\epsilon), \quad -1 < \epsilon < \infty,$$

such $E_\xi$ $E_\xi$ is an increasing mapping of $(0,\infty)(0,\infty)$onto itself. Moreover, if $R \in (0,\infty], R \in (0,\infty]$, then the following relations hold.

$$\|(1+\epsilon)^\xi \chi_{(0,a)}(1+\epsilon)\|_{L^A(0,R)} = \frac{a^\xi}{E_\xi^{-1}\left(\frac{1}{a}\right)}, a \in (0,R), \xi > 0, \qquad (3.2)$$

$$\|(1+\epsilon)^\xi \chi_{(a,\infty)}(1+\epsilon)\|_{L^A(0,\infty)} = \frac{a^\xi}{E_\xi^{-1}\left(\frac{1}{a}\right)}, a \in (0,\infty), \xi < 0. \ (3.3)$$

If, in addition, $\epsilon \in (0,R)\epsilon \in (0,R)$ and if $\xi < 0\xi < 0$ then

$$\|(1+\epsilon)^\xi \chi_{(a,R)}(1+\epsilon)\|_{L^A(0,R)} \simeq \|(1+\epsilon)^\xi \chi_{(a,\infty)}(1+\epsilon)\|_{L^A(0,\infty)}, a \in (0, R-\varepsilon). (3.4)$$

**Proof.** Assume (3.1). By change of variables $1-\epsilon \mapsto (1-\epsilon^2)$ $1-\epsilon \mapsto (1-\epsilon^2)$ we have

$$E_\xi(1+\epsilon) = |\xi|^{-1} \int_0^1 A\left((1-\epsilon^2)\right)(1-\epsilon)^{\frac{1}{\xi}-1} d(1-\epsilon), \quad -1 < \epsilon < \infty,$$

hence $E_\xi$ $E_\xi$ is increasing. By definition of the Luxemburg norm, we have

$$\|(1+\epsilon)^\xi \chi_{(0,a)}(1+\epsilon)\|_{L^A(0,R)} = \inf\left\{\lambda > 0, \int_0^a A\left(\frac{(1+\epsilon)^\xi}{\lambda}\right) d(1+\epsilon) \leq 1\right\}.$$

Next, by change of variables we get for $\xi < 0\xi < 0$

$$\left\|(1+\epsilon)^{\xi}\chi_{(0,a)}(1+\epsilon)\right\|_{L^{A}(0,R)}$$

$$=\inf\left\{\lambda>0,\frac{\lambda^{\frac{1}{\xi}}}{\xi}\int_{0}^{\frac{a^{\xi}}{\lambda}}A(1-\epsilon)(1-\epsilon)^{\frac{1}{\xi}-1}d(1+\epsilon)\leq 1\right\}$$

$$=\inf\left\{\lambda>0,a\,E_{\xi}\left(\frac{a^{\xi}}{\lambda}\right)\leq 1\right\}=\frac{a^{\xi}}{E_{\xi}^{-1}\left(\frac{1}{a}\right)}.$$

This proves the part (3.2).The proof of the relation (3.3) can be done in an analogous way and we omit it. It remains to prove the (3.4). Clearly,

$$\left\|(1+\epsilon)^{\xi}\chi_{(a,\infty)}(1+\epsilon)\right\|_{L^{A}(0,\infty)}\geq\left\|(1+\epsilon)^{\xi}\chi_{(a,R)}(1+\epsilon)\right\|_{L^{A}(0,\infty)}=\left\|(1+\epsilon)^{\xi}\chi_{(a,R)}(1+\epsilon)\right\|_{L^{A}(0,R)}$$

by the monotonicity of the norm. On the other hand, we have by the triangle inequality

$$\left\|(1+\epsilon)^{\xi}\chi_{(a,\infty)}(1+\epsilon)\right\|_{L^{A}(0,\infty)}\leq\left\|(1+\epsilon)^{\xi}\chi_{(a,R)}(1+\epsilon)\right\|_{L^{A}(0,R)}+\left\|(1+\epsilon)^{\xi}\chi_{(R,\infty)}(1+\epsilon)\right\|_{L^{A}(0,\infty)}.$$

$$\left\|(1+\epsilon)^{\xi}\chi_{(a,\infty)}(1+\epsilon)\right\|_{L^{A}(0,\infty)}\leq\left\|(1+\epsilon)^{\xi}\chi_{(a,R)}(1+\epsilon)\right\|_{L^{A}(0,R)}+\left\|(1+\epsilon)^{\xi}\chi_{(R,\infty)}(1+\epsilon)\right\|_{L^{A}(0,\infty)}.$$

Using (3.3), the term $\left\|(1+\epsilon)^{\xi}\chi_{(R,\infty)}(1+\epsilon)\right\|_{L^{A}(0,\infty)}$

$\left\|(1+\epsilon)^{\xi}\chi_{(R,\infty)}(1+\epsilon)\right\|_{L^{A}(0,\infty)}$ equals $\frac{R^{\xi}}{E_{\xi}^{-1}\left(\frac{1}{R}\right)}\frac{R^{\xi}}{E_{\xi}^{-1}\left(\frac{1}{R}\right)}$ since $\xi<0$. $\xi<0$. Thanks to the assumptions, this quantity is finite, say $(1+\epsilon)(1+\epsilon)$. The term

$\left\|(1+\epsilon)^{\xi}\chi_{(a,R)}(1+\epsilon)\right\|_{L^{A}(0,R)}\left\|(1+\epsilon)^{\xi}\chi_{(a,R)}(1+\epsilon)\right\|_{L^{A}(0,R)}$ is a decreasing function of the variable $aa$, positive on $(0,R)(0,R)$ and vanishing at $RR$. Hence for every $\epsilon \in (0,R)\epsilon \in (0,R)$there exists a constant $C\ C$ such that $1+\epsilon \leq C\left\|(1+\epsilon)^{\xi}\chi_{(a,R)}(1+\epsilon)\right\|_{L^A(0,R)}$

$1+\epsilon \leq C\left\|(1+\epsilon)^{\xi}\chi_{(a,R)}(1+\epsilon)\right\|_{L^A(0,R)}$, $a \in (0,R-\epsilon)$. $a \in (0,R-\epsilon)$. For those $aa$ we conclude that $\left\|(1+\epsilon)^{\xi}\chi_{(a,\infty)}(1+\epsilon)\right\|_{L^A(0,\infty)} \leq (C+1)\left\|(1+\epsilon)^{\xi}\chi_{(a,R)}(1+\epsilon)\right\|_{L^A(0,R)}.$

**Lemma 3.2.** Let $0 \leq \epsilon < 1, \epsilon \geq 1, \epsilon \geq \frac{1}{2} 0 \leq \epsilon < 1, \epsilon \geq 1, \epsilon \geq \frac{1}{2}$ and let $\varphi\ \varphi$ be a quasiconcave function on $(0,1).(0,1)$. We define $\overline{\varphi}(1+\epsilon) = (1+\epsilon)^{-\epsilon(1+\epsilon)} \sup_{\frac{1}{2}>\epsilon>0} \varphi(1-\epsilon)(1-\epsilon)^{-\epsilon(1+\epsilon)}, 0<\epsilon<1, \overline{\varphi}(0)=0.$

Then $\overline{\varphi}(1+\epsilon)\overline{\varphi}(1+\epsilon)$ and $\overline{\varphi}\left((1+\epsilon)^{\frac{1}{1+\epsilon}}\right)(1+\epsilon)^{(1-\epsilon)}$ $\overline{\varphi}\left((1+\epsilon)^{\frac{1}{1+\epsilon}}\right)(1+\epsilon)^{(1-\epsilon)}$ are quasiconcave.

**Proof.** Since $\varphi\varphi$ is nondecreasing , we have for every $0<\epsilon<1$, $0<\epsilon<1$,

$$\overline{\varphi}(1-\epsilon) = (1-\epsilon)^{-\epsilon(1+\epsilon)} \sup_{0<\epsilon<\frac{1}{2}} \varphi(1+2\epsilon)$$

$$= (1-\epsilon)^{-\epsilon(1+\epsilon)} \sup_{0<\epsilon<\frac{1}{2}} \varphi(1+2\epsilon) \sup_{\max\{1+2\epsilon,1-\epsilon\}<1-\epsilon<1} (1-\epsilon)^{-\epsilon(1+\epsilon)}$$

$$= (1-\epsilon)^{-\epsilon(1+\epsilon)} \sup_{0<\epsilon<\frac{1}{2}} \varphi(1+2\epsilon)\min\{(1-\epsilon)^{-\epsilon(1+\epsilon)},(1+2\epsilon)^{-\epsilon(1+\epsilon)}\}$$

$$= \sup_{0<\epsilon<\frac{1}{2}} \varphi(1+2\epsilon)\min\left\{1,\left(\frac{1-\epsilon}{1+2\epsilon}\right)^{-\epsilon(1+\epsilon)}\right\},$$

hence $\overline{\varphi}\ \overline{\varphi}$ is nondecreasing. Next, by definition, we have

$$\frac{\overline{\varphi}(1-\epsilon)}{1-\epsilon} = (1-\epsilon)^{-\epsilon(1+\epsilon)-1} \sup_{\frac{1}{2}>\epsilon>0} \varphi(1-\epsilon)(1-\epsilon)^{-\epsilon(1+\epsilon)},\ 0<\epsilon<1.\varphi(1+2\epsilon)$$

The function $(1-\epsilon)^{(-\epsilon^2+\epsilon-1)}(1-\epsilon)^{(-\epsilon^2+\epsilon-1)}$ is nonincreasing since the exponent $(\epsilon^2-\epsilon+1)(\epsilon^2-\epsilon+1)$is nonnegative by the assumptions of the lemma. Hence $\frac{\overline{\varphi}(1-\epsilon)}{1-\epsilon}\frac{\overline{\varphi}(1-\epsilon)}{1-\epsilon}$ is nonincreasing on $(0,1).(0,1)$. The function $\overline{\varphi}\left((1-\epsilon)^{\frac{1}{1+\epsilon}}\right)(1-\epsilon)^{1-\epsilon}$ $\overline{\varphi}\left((1-\epsilon)^{\frac{1}{1+\epsilon}}\right)(1-\epsilon)^{1-\epsilon}$ is increasing as a composition of nondecreasing functions multiplied by increasing function $(1+\epsilon)^{(1-\epsilon)}.(1+\epsilon)^{(1-\epsilon)}$. Next the expression

$$\frac{\overline{\varphi}\left((1-\epsilon)^{\frac{1}{1+\epsilon}}\right)(1-\epsilon)^{1-\epsilon}}{1-\epsilon} = \sup_{\frac{1}{2}>\epsilon>0} \varphi\left((1-\epsilon)^{\frac{1}{1+\epsilon}}\right)(1-\epsilon)^{(-\epsilon)}, 0<\epsilon<1,$$

is nonincreasing. The rest is trivial.

**Lemma 3.3.** Let $uu$ be a quasiconcave, right continuous at origin and strictly increasing function on $[0,1)\ [0,1)$ such that $\lim_{1+\epsilon\to 0^+}\frac{u(1+\epsilon)}{1+\epsilon} = 0. \lim_{1+\epsilon\to 0^+}\frac{u(1+\epsilon)}{1+\epsilon} = 0.$

Then there exists a Young function $BB$ such that the fundamental function of the Orlicz space $L^B(0,1)L^B(0,1)$ is equivalent to $u\,u$ on $[0,1)[0,1)$

. Moreover $\tilde{B}^{-1}(1+\epsilon) \simeq (1+\epsilon)\, u\left(\frac{1}{1+\epsilon}\right), 0 < \epsilon < \infty$,

$\tilde{B}^{-1}(1+\epsilon) \simeq (1+\epsilon)\, u\left(\frac{1}{1+\epsilon}\right), 0 < \epsilon < \infty$,

where $\tilde{B}$ $\tilde{B}$ is the complementary Young function to $BB$.

**Proof.** We can assume without loss of generality that $u(1) = 1$. $u(1) = 1$. Then by continuity of $uu$ we have $u(0,1) = (0,1)$. $u(0,1) = (0,1)$. Let us define

$$b(1+\epsilon) = \begin{cases} \dfrac{1}{(1+\epsilon)u^{-1}\left(\dfrac{1}{1+\epsilon}\right)}, & 0 < \epsilon < \infty, \\ 1-\epsilon, & 0 \le \epsilon \le 1, \end{cases}$$

and $B(1+\epsilon) = \int_0^{1+\epsilon} b(1+\epsilon)d(1+\epsilon), \quad -1 \le \epsilon < \infty$ $B(1+\epsilon) = \int_0^{1+\epsilon} b(1+\epsilon)d(1+\epsilon), \quad -1 \le \epsilon < \infty$.

We claim that $BB$ is a Young function.The properties (i) and (ii) from the definition of Young function are clear. Let us prove that $b$ $b$ is nondecreasing.The function $\frac{u(1-\epsilon)}{1-\epsilon}$ $\frac{u(1-\epsilon)}{1-\epsilon}$ is nonincreasing and $uu$ itself is increasing, hence $\frac{1+\epsilon}{u^{-1}(1-\epsilon)}$ $\frac{1+\epsilon}{u^{-1}(1-\epsilon)}$ is nonincreasing and therefore

$b(1+\epsilon) = \frac{1}{(1+\epsilon)u^{-1}\left(\frac{1}{1+\epsilon}\right)}$ $b(1+\epsilon) = \frac{1}{(1+\epsilon)u^{-1}\left(\frac{1}{1+\epsilon}\right)}$ is nondecreasing on $(1,\infty)(1,\infty)$ and also (trivially) on $[0,1]$. $[0,1]$. It remains to show that $\lim_{1+\epsilon\to\infty} b(1+\epsilon) = \infty$. $\lim_{1+\epsilon\to\infty} b(1+\epsilon) = \infty$. Indeed,

$$\lim_{1+\epsilon\to\infty} b(1+\epsilon) = \lim_{1-\epsilon\to 0^+} \frac{1-\epsilon}{u^{-1}(1-\epsilon)} = \lim_{1-\epsilon\to 0^+} \frac{u(1-\epsilon)}{1-\epsilon} = \infty.$$

Now, since $BB$ is a Young function, we have that

$$B(1-\epsilon) \le b(1-\epsilon^2) \le B(2-2\epsilon), -1 \le \epsilon < \infty.$$

$B(1-\epsilon) \le b(1-\epsilon^2) \le B(2-2\epsilon), -1 \le \epsilon < \infty$. It follows by definition of $bb$ that $B(1+\epsilon) \le \frac{1}{u^{-1}\left(\frac{1}{1+\epsilon}\right)} \le B(2+2\epsilon), 0 < \epsilon < \infty$

$B(1+\epsilon) \le \frac{1}{u^{-1}\left(\frac{1}{1+\epsilon}\right)} \le B(2+2\epsilon), 0 < \epsilon < \infty.$

Applying the increasing function $B^{-1}, B^{-1}$, we get $1+\epsilon \le B^{-1}\left(\frac{1}{u^{-1}\left(\frac{1}{1+\epsilon}\right)}\right) \le 2-2\epsilon,$

$1+\epsilon \le B^{-1}\left(\frac{1}{u^{-1}\left(\frac{1}{\cdot}\right)}\right) \le 2-2\epsilon,$

$0 < \epsilon < \infty, 0 < \epsilon < \infty$, that is, taking reciprocal values and

$$1-\epsilon \mapsto \frac{1}{1+\epsilon}, \frac{1+\epsilon}{2} \le \frac{1}{B^{-1}\left(\frac{1}{u^{-1}\left(\frac{1}{1-\epsilon}\right)}\right)} \le 1+\epsilon, \quad 0 < \epsilon < \infty$$

$1-\epsilon \mapsto \frac{1}{1+\epsilon}, \frac{1+\epsilon}{2} \le \frac{1}{B^{-1}\left(\frac{1}{u^{-1}\left(\frac{1}{1-\epsilon}\right)}\right)} \le 1+\epsilon, \quad 0 < \epsilon < \infty$

. Finally, since $uu^{u^{-1}}$ is increasing on $(0,1)(0,1)$ and $u(0,1) = (0,1)\, u(0,1) = (0,1)$ this implies

$$\frac{u((1-\epsilon))}{2} \le \frac{1}{B^{-1}\left(\frac{1}{(1-\epsilon)}\right)} \le u((1-\epsilon)), \quad \epsilon < 1.$$

$\frac{u((1-\epsilon))}{2} \le \frac{1}{B^{-1}\left(\frac{1}{(1-\epsilon)}\right)} \le u((1-\epsilon)), \quad \epsilon < 1.$ Hence by the definition of the fundamental function for the Luxemburg norm we conclude that $\varphi_{L^B}(1-\epsilon) \simeq u(1-\epsilon), \quad 0 < \epsilon < 1.$ $\varphi_{L^B}(1-\epsilon) \simeq u(1-\epsilon), \quad 0 < \epsilon < 1.$

In addition $\tilde{B}^{-1}(1-\epsilon) \simeq \frac{1-\epsilon}{B^{-1}(1-\epsilon)} \simeq (1-\epsilon)u\left(\frac{1}{1-\epsilon}\right), 0 < \epsilon < 1.$ $\tilde{B}^{-1}(1-\epsilon) \simeq \frac{1-\epsilon}{B^{-1}(1-\epsilon)} \simeq (1-\epsilon)u\left(\frac{1}{1-\epsilon}\right), 0 < \epsilon < 1.$ The proof is complete.

The following proposition enables us to reduce an embedding to a Lorentz endpoint spaces only to testing on characteristic functions. The idea of this statement is based on [2, Theorem 7], where the Lorentz space $L^{(1+\epsilon,1)}(\Omega)L^{(1+\epsilon,1)}(\Omega)$ occurs as a target space, nonethelesss the proof also works for any Lorentz endpoint space. For the sake of completeness, we show also the proof here (see [20]).

**Proposition 3.4.** Let $Y(0,1)Y(0,1)$ be a Banach function space and $\Lambda(0,1)\Lambda(0,1)$ be a Lorentz endpoint space over $(0,1)(0,1)$. Suppose that $TT$ is a sublinear operator mapping $\Lambda(0,1)\Lambda(0,1)$ to $\mathrm{Y}(0,1)\mathrm{Y}(0,1)$ and satisfying

$\|T_{\chi_E}\|_{\mathrm{Y}(0,1)} \lesssim \|\chi_E\|_{\Lambda(0,1)}$ (3.5)

for every measurable set $E \subseteq (0,1).E \subseteq (0,1)$. Then

$\|T\sum f_j\|_{\mathrm{Y}(0,1)} \lesssim \sum\|f_j\|_{\Lambda(0,1)}$ $\|T\sum f_j\|_{\mathrm{Y}(0,1)} \lesssim \sum\|f_j\|_{\Lambda(0,1)}$ for every $f_j \in \Lambda(0,1).f_j \in \Lambda(0,1)$.

**Proof.** Let $f_j f_j$ be a simple nonnegative functions on $(0,1).(0,1)$. Thus $f_j f_j$ can be written as a finite sum $f_j = \sum_j \lambda_j \chi_{E_j}$ $f_j = \sum_j \lambda_j \chi_{E_j}$, where $\lambda_j \lambda_j$ are positive real numbers and the sets $E_j$ $E_j$ are measurable subsets of $(0,1)(0,1)$ satisfying $E_1 \subseteq E_2 \subseteq \cdots.$ $E_1 \subseteq E_2 \subseteq \cdots.$ Then, as readily seen, we have $f_j^* = \sum_j \lambda_j \chi_{E_j}^*$

$f_j^* = \sum_j \lambda_j \chi_{E_j}^*$. Let $\varphi\ \varphi$ be a fundamental function of $\Lambda(0,1)$. $\Lambda(0,1)$. By the definition of the Lorentz norm we have

$$\sum \|f_j\|_{\Lambda(0,1)} = \int_0^1 \sum f_j^* \, d_\varphi = \int_0^1 \sum_j \lambda_j \chi_{E_j}^* \, d_\varphi = \sum_j \lambda_j \int_0^1 \chi_{E_j}^* \, d_\varphi = \sum_j \lambda_j \left\|\chi_{E_j}\right\|_{\Lambda(0,1)}.$$

On account of the sublinearity of $T\ T$ we have $\left|T\left(\sum f_j\right)\right| \le \sum_j \lambda_j \left|T_{\chi_{E_j}}\right|, \left|T\left(\sum f_j\right)\right| \le \sum_j \lambda_j \left|T_{\chi_{E_j}}\right|,$ and consequectly by (3.5) and by axioms (P1) and (P2) we obtain

$$\left\|T\sum f_j\right\|_{Y(0,1)} \le \sum_j \lambda_j \left\|T_{\chi_{E_j}}\right\|_{Y(0,1)} \lesssim \sum_j \lambda_j \left\|\chi_{E_j}\right\|_{\Lambda(0,1)} = \sum \left\|f_j\right\|_{\Lambda(0,1)}.$$

$$\left\|T\sum f_j\right\|_{Y(0,1)} \le \sum_j \lambda_j \left\|T_{\chi_{E_j}}\right\|_{Y(0,1)} \lesssim \sum_j \lambda_j \left\|\chi_{E_j}\right\|_{\Lambda(0,1)} = \sum \left\|f_j\right\|_{\Lambda(0,1)}.$$

Now if $f_j f_j$ is simple but no longer nonnegative, we use the same for the positive part of $f_j\ f_j$ and for the negative part of $f_j\ f_j$ .

Suppose that $f_j\ f_j$ is an arbitrary functions in $\Lambda(0,1)\Lambda(0,1)$ and let $(f_j)_{1+2\epsilon}(f_j)_{1+2\epsilon}$ be a sequence of simple integrable functions converging to $f_i\ f_i$ in $\Lambda(0,1)\Lambda(0,1)$ Then

$$\left\|\sum T(f_j)_{1+2\epsilon} - T\left((f_j)_m\right)\right\|_{Y(0,1)}$$

$$\le \left\|\sum T\left((f_j)_{1+2\epsilon} - (f_j)_m\right)\right\|_{Y(0,1)} \lesssim \left\|\sum (f_j)_{1+2\epsilon} - (f_j)_m\right\|_{\Lambda(0,1)},$$

and $T\,(f_j)_{1+2\epsilon}\ T\,(f_j)_{1+2\epsilon}$ is Cauchy, hence convergent in $Y(0,1)$. $Y(0,1)$. Since limits are unique in $Y(0,1), Y(0,1)$, it follows that $\lim T\,(f_j)_{1+2\epsilon} = Tf_j\ \lim T\,(f_j)_{1+2\epsilon} = Tf_j$ and

$$\left\|T\sum f_j\right\|_{Y(0,1)} = \lim\left\|T\sum (f_j)_{1+2\epsilon}\right\|_{Y(0,1)} \lesssim \lim\sum\left\|(f_j)_{1+2\epsilon}\right\|_{\Lambda(0,1)} = \sum\|f_j\|_{\Lambda(0,1)}$$

as we wished to show. Next proposition provides the optimal r.i. range space for the operator $H_{1-\epsilon}^{1+\epsilon}H_{1-\epsilon}^{1+\epsilon}$ and a given r.i. domain space.The proof can be obtained by simple modification of the proof of [11, Theorem 4.5], where $\epsilon = 0$ $\epsilon = 0$ or $\epsilon = -\frac{1}{2}\epsilon = -\frac{1}{2}$ and therefore is omitted (see [20 ]).

**Proposition3.5.**LetX(0,1)X(0,1)beanr.i.space,$0 \le \epsilon < 1,\ \epsilon \ge 0$ $0 \le \epsilon < 1,\ \epsilon \ge 0$and$\ge \frac{1}{2} \ge \frac{1}{2}$.Then

$$Y'(0,1) := \left\{f_j \in \mathcal{M}(0,1), \sum\|f_j\|_{Y'(0,1)} := \left\|\sum (H_{1-\epsilon}^{1+\epsilon})' f_j^*\right\|_{X'(0,1)} \le \infty\right\}$$

is an r.i. space, such that the associate space Y(0,1)Y(0,1) is the smallest space among r.i. spaces rendering $H_{1-\epsilon}^{1+\epsilon}: X(0,1) \to Y(0,1)$ $H_{1-\epsilon}^{1+\epsilon}: X(0,1) \to Y(0,1)$ true.The construction of the optimal r.i. domain for $H_{1-\epsilon}^{1+\epsilon}H_{1-\epsilon}^{1+\epsilon}$ and a given r.i. range space is similar to that in [16, Theorem 3.3], as well as its proof, needing only trivial modifications. The fact that $\mu_{\sum f_j} = \mu_h \mu_{\sum f_j} = \mu_h$ is denoted by $f_j \sim h.f_j \sim h$.

**Proposition 3.6.**Let Y(0,1)Y(0,1) be an r.i. space such that $Y(0,1) \hookrightarrow L^{\left(\frac{1}{-\epsilon(1+\epsilon)},1\right)}(0,1)$. $Y(0,1) \hookrightarrow L^{\left(\frac{1}{-\epsilon(1+\epsilon)},1\right)}(0,1)$. Then

$$X(0,1) := \left\{f_j \in \mathcal{M}(0,1), \sum\|f_j\|_{X(0,1)} := \sup_{f_j \sim h}\|H_{1-\epsilon}^{1+\epsilon} h\|_{Y(0,1)} < \infty\right\}$$

is the largest r.i. space satisfying $H_{1-\epsilon}^{1+\epsilon}: X(0,1) \to Y(0,1)$. $H_{1-\epsilon}^{1+\epsilon}: X(0,1) \to Y(0,1)$.

**Remark 3.7.** Now, under the same assumptions on $(1-\epsilon)(1-\epsilon)$ and $(1+\epsilon)\,(1+\epsilon)$ as in Proposition 3.5. one can readily calculate the optimal endpoint estimates

$$H_{1-\epsilon}^{1+\epsilon}: L^1(0,1) \to L^{\frac{1}{-\epsilon(1+\epsilon)},1}(0,1) \qquad (3.6)$$

a n d

$$H_{1-\epsilon}^{1+\epsilon}: L^{\frac{1}{1-\epsilon},1}(0,1) \to L^{\infty}(0,1). \qquad (3.7)$$

The relation (3.6) shows that the assumption in proposition 3.6 cause no loss of generality.
Let us also discuss the assumption on fundamental function

$$\sup_{0<\epsilon<1} \varphi\left((1+\epsilon)^{\frac{1}{1+\epsilon}}\right)(1+\epsilon)^{-\epsilon} = \infty$$

$\sup_{0<\epsilon<1} \varphi\left((1+\epsilon)^{\frac{1}{1+\epsilon}}\right)(1+\epsilon)^{-\epsilon} = \infty$ in Theorem A and B. If this condition is not satisfied, then $\varphi(1+\epsilon) \leq C(1+\epsilon)^{-\epsilon(1+\epsilon)}, 0<\epsilon<1,$ $\varphi(1+\epsilon) \leq C(1+\epsilon)^{-\epsilon(1+\epsilon)}, 0<\epsilon<1,$ for some $C>0,$ $C>0,$ which is equivalent to $L^{\frac{1}{-\epsilon(1+\epsilon)},\infty}(0,1) \subseteq M_{\varphi}(0,1),$ $L^{\frac{1}{-\epsilon(1+\epsilon)},\infty}(0,1) \subseteq M_{\varphi}(0,1),$ hence, thanks to (3.6) also to $H_{1-\epsilon}^{1+\epsilon}: L^1(0,1) \to M_{\varphi}(0,1).H_{1-\epsilon}^{1+\epsilon}: L^1(0,1) \to M_{\varphi}(0,1).$ Since $L^1(0,1)L^1(0,1)$ is the largest r.i. space, we can see that this considered assumption cause no relevant restriction to target spaces.

**Proof of Theorem C.** We first prove the inequality" $\gtrsim\gtrsim$". Let $(1-\epsilon)\,(1-\epsilon)$ and $(1+\epsilon)(1+\epsilon)$ be as in the theorem and let us se t $\psi(1+\epsilon)=(1+\epsilon)\sup_{1>2\epsilon>0}\varphi\left((1+\epsilon)^{\frac{1}{1+\epsilon}}\right)(1+\epsilon)^{-\epsilon},\quad 0<\epsilon<1.$

$\psi(1+\epsilon)=(1+\epsilon)\sup_{1>2\epsilon>0}\varphi\left((1+\epsilon)^{\frac{1}{1+\epsilon}}\right)(1+\epsilon)^{-\epsilon},\quad 0<\epsilon<1.$

Then, by Lemma 3.2, $\psi(1+\epsilon)$ $\psi(1+\epsilon)$ is quasiconcave function on $(0,1)(0,1)$ and

$\psi(1+\epsilon)\geq(1+\epsilon)^{1-\epsilon}\varphi\left((1+\epsilon)^{\frac{1}{1+\epsilon}}\right)$

$\psi(1+\epsilon)\geq(1+\epsilon)^{1-\epsilon}\varphi\left((1+\epsilon)^{\frac{1}{1+\epsilon}}\right)$ for $,0<\epsilon<1.$ $,0<\epsilon<1.$ We claim that $\psi\psi$ is up to equivalence the smallest function with this property. Indeed, let $\eta(1+\epsilon)\eta(1+\epsilon)$ be a quasiconcave function on $[0,1)[0,1)$ and

$\eta(1+\epsilon)\geq(1+\epsilon)^{1-\epsilon}\varphi\left((1+\epsilon)^{\frac{1}{1+\epsilon}}\right)$

$\eta(1+\epsilon)\geq(1+\epsilon)^{1-\epsilon}\varphi\left((1+\epsilon)^{\frac{1}{1+\epsilon}}\right)$

for $,0<\epsilon<1.$ $,0<\epsilon<1.$ Then

$\varphi\left((1+\epsilon)^{\frac{1}{1+\epsilon}}\right)(1+\epsilon)^{-\epsilon}\leq\frac{\eta(1+\epsilon)}{1+\epsilon},\ 0<\epsilon<1, \sup_{\frac{1}{2}>\epsilon>0}\varphi\left((1+\epsilon)^{\frac{1}{1+\epsilon}}\right)(1+\epsilon)^{-\epsilon}\leq \sup_{\frac{1}{2}>\epsilon>0}\frac{\eta(1+\epsilon)}{1+\epsilon},$

$\varphi\left((1+\epsilon)^{\frac{1}{1+\epsilon}}\right)(1+\epsilon)^{-\epsilon}\leq\frac{\eta(1+\epsilon)}{1+\epsilon},\ 0<\epsilon<1, \sup_{\frac{1}{2}>\epsilon>0}\varphi\left((1+\epsilon)^{\frac{1}{1+\epsilon}}\right)(1+\epsilon)^{-\epsilon}\leq \sup_{\frac{1}{2}>\epsilon>0}\frac{\eta(1+\epsilon)}{1+\epsilon},$

$0<\epsilon<1$ $0<\epsilon<1$.The right hand side of the last inequality equals $\frac{\eta(1+\epsilon)\eta(1+\epsilon)}{1+\epsilon\quad 1+\epsilon}$ by quasiconcavity of $\eta.\eta$. Then multiplying by $(1+\epsilon)(1+\epsilon)$ gives that $\psi(1+\epsilon)\leq\eta(1+\epsilon)$

$\psi(1+\epsilon) \leq \eta(1+\epsilon)$for $0<\epsilon<1,\ 0<\epsilon<1,.$ Now by Proposition 3.6 we have

$$\varphi_X(1+\epsilon) = \sup_{h\sim\chi_{(0,1+\epsilon)}} \left\| \int_{(1+\epsilon)^{1+\epsilon}}^{1} (1-\epsilon)^{-\epsilon} h(1-\epsilon) d(1-\epsilon) \right\|_{M(0,1)}$$

$$\geq \left\| \int_{(1+\epsilon)^{1+\epsilon}}^{1} (1-\epsilon)^{-\epsilon} \chi_{(0,1+\epsilon)}(1-\epsilon) d(1-\epsilon) \right\|_{M(0,1)}$$

$$= \left\| \chi_{\left(0,(1+\epsilon)^{\frac{1}{1+\epsilon}}\right)}(1+\epsilon) \int_{1+\epsilon^{(1+\epsilon)}}^{1+\epsilon} (1-\epsilon)^{-\epsilon} d(1-\epsilon) \right\|_{M(0,1)}$$

$$\simeq \left\| \chi_{\left(0,(1+\epsilon)^{\frac{1}{1+\epsilon}}\right)}(1+\epsilon)\left((1+\epsilon)^{1-\epsilon} - (1+\epsilon)^{1-\epsilon^2}\right) \right\|_{M(0,1)}$$

$$\geq \left\| \chi_{\left(0,\frac{(1+\epsilon)^{\frac{1}{1+\epsilon}}}{2}\right)}(1+\epsilon)\left((1+\epsilon)^{1-\epsilon} - (1+\epsilon)^{1-\epsilon}2^{\epsilon^2-1}\right) \right\|_{M(0,1)}$$

$$\simeq (1+\epsilon)^{1-\epsilon} \left\| \chi_{\left(0,\frac{(1+\epsilon)^{\frac{1}{1+\epsilon}}}{2}\right)}(1+\epsilon) \right\|_{M(0,1)} \simeq$$

$$(1+\epsilon)^{1-\epsilon} \left\| \chi_{\left(0,(1+\epsilon)^{\frac{1}{1+\epsilon}}\right)}(1+\epsilon) \right\|_{M(0,1)} = (1+\epsilon)^{1-\epsilon}\varphi\left((1+\epsilon)^{\frac{1}{1+\epsilon}}\right).$$

Hence $\varphi_X(1+\epsilon) \gtrsim (1+\epsilon)^{1-\epsilon}\varphi\left((1+\epsilon)^{\frac{1}{1+\epsilon}}\right)$

$\varphi_X(1+\epsilon) \gtrsim (1+\epsilon)^{1-\epsilon}\varphi\left((1+\epsilon)^{\frac{1}{1+\epsilon}}\right)$ and by the claim

$\psi(1+\epsilon) \lesssim \varphi_X(1+\epsilon).\psi(1+\epsilon) \lesssim \varphi_X(1+\epsilon)$. Let us focus on the inequality" $\lesssim\lesssim$". Let $0<\epsilon<\frac{1}{2}0<\epsilon<\frac{1}{2}$ then

$$\varphi_X(1+\epsilon) = \sup_{h\sim\chi(0,1+\epsilon)} \left\| \int_{(1+\epsilon)^{1+\epsilon}}^{1} (1-\epsilon)^{-\epsilon} h(1-\epsilon) d(1-\epsilon) \right\|_{M(0,1)}$$

$$= \sup_{h\sim\chi(0,1+\epsilon)} \sup_{0<-\epsilon<1} \left( \int_{(1+2\epsilon)^{1+\epsilon}}^{1} (1-\epsilon)^{-\epsilon} h(1-\epsilon) d(1-\epsilon) \right)^{**} (1+\epsilon)\varphi(1+\epsilon)$$

$$= \sup_{h\sim\chi(0,1+\epsilon)} \sup_{0<\epsilon<1} \frac{\varphi(1+\epsilon)}{1+\epsilon} \int_0^{1+\epsilon} \int_{(1+2\epsilon)^{1+\epsilon}}^{1} (1-\epsilon)^{-\epsilon} h(1-\epsilon) d(1-\epsilon)\, dr$$

$$= \sup_{h\sim\chi(0,1+\epsilon)} \sup_{0<\epsilon<1} \frac{\varphi(1+\epsilon)}{1+\epsilon} \int_0^{1} (1-\epsilon)^{-\epsilon} h(1-\epsilon) \int_0^{\min\left\{(1-\epsilon)^{\frac{1}{1+\epsilon}},1+\epsilon\right\}} d(1+2\epsilon) d(1-\epsilon)$$

$$= \sup_{0<\epsilon<1} \sup_{h\sim\chi(0,1+\epsilon)} \frac{\varphi(1+\epsilon)}{1+\epsilon} \left( \int_0^{(1+\epsilon)^{1+\epsilon}} (1-\epsilon)^{\frac{\epsilon^2+\epsilon-1}{1+\epsilon}} h(1-\epsilon) d(1-\epsilon) + (1+\epsilon) \int_{(1+\epsilon)^{1+\epsilon}}^{1} (1-\epsilon)^{-\epsilon} h(1-\epsilon) d(1-\epsilon) \right)$$

$$= \sup_{0<\epsilon<1} \sup_{h\sim\chi(0,1-\epsilon)\ (1+\epsilon)^{1+\epsilon}<2(1-\epsilon)<1} \frac{\varphi(1+\epsilon)}{1+\epsilon} \left( \int_{(1-\epsilon)}^{(1+\epsilon)^{1+\epsilon}} (1-\epsilon)^{\frac{\epsilon^2+\epsilon-1}{1+\epsilon}} d(1-\epsilon) + (1+\epsilon) \int_{(1+\epsilon)^{1+\epsilon}}^{2} (1-\epsilon)^{-\epsilon} d(1-\epsilon) \right).$$

Denote

$$V(1+\epsilon,1-\epsilon,1+\epsilon)$$
$$=\frac{\varphi(1+\epsilon)}{1+\epsilon}\left(\int_{(1-\epsilon)}^{(1+\epsilon)^{1+\epsilon}}(1-\epsilon)^{\frac{\epsilon^2+\epsilon-1}{1+\epsilon}}d(1-\epsilon)\right.$$
$$\left.+(1+\epsilon)\int_{(1+\epsilon)^{1+\epsilon}}^{2}(1-\epsilon)^{-\epsilon}d(1-\epsilon)\right).$$

We split the area over which the supremum is taken into three disjoint regions namely

$$\varphi_X(1+\epsilon)\leq \sup_{0<1+\epsilon<(1+\epsilon)^{\frac{1}{1+\epsilon}}\ 0<1-\epsilon<(1+\epsilon)^{1+\epsilon}} V(1+\epsilon,1-\epsilon,1+\epsilon)$$
$$+\sup_{(1+\epsilon)^{\frac{1}{1+\epsilon}}<1+\epsilon<(-\epsilon)^{\frac{1}{1+\epsilon}}(1+\epsilon)^{1+\epsilon}-(1+\epsilon)<1-\epsilon<(1+\epsilon)^{1+\epsilon}} V(1+\epsilon,1-\epsilon,1+\epsilon)$$
$$+\sup_{(-\epsilon)^{\frac{1}{1+\epsilon}}<1+\epsilon<1(1+\epsilon)^{1+\epsilon}+1} V(1+\epsilon,1-\epsilon,1+\epsilon).$$

Now

$$\sup_{0<1+\epsilon<(1-\epsilon)^{\frac{1}{1+\epsilon}}0<1-\epsilon<(1+\epsilon)^{1+\epsilon}} V(1+\epsilon,1-\epsilon,1+\epsilon)$$
$$\leq \sup_{0<1+\epsilon<(1-\epsilon)^{\frac{1}{1+\epsilon}}}\frac{\varphi(1+\epsilon)}{1+\epsilon}\left(\int_{(1-\epsilon)}^{(1+\epsilon)^{1+\epsilon}}(1-\epsilon)^{\frac{\epsilon^2+\epsilon-1}{1+\epsilon}}d(1-\epsilon)\right.$$
$$\left.+(1+\epsilon)\int_{(1+\epsilon)^{1+\epsilon}}^{(1+\epsilon)^{1+\epsilon}+(1-\epsilon)}(1-\epsilon)^{-\epsilon}d(1-\epsilon)\right)$$
$$\leq \sup_{0<1+\epsilon<(1-\epsilon)^{\frac{1}{1+\epsilon}}}\frac{\varphi(1+\epsilon)}{1+\epsilon}\left(\int_{0}^{(1+\epsilon)^{1+\epsilon}}(1+\epsilon)^{-\epsilon^2-\epsilon+1}d(1-\epsilon)\right.$$
$$\left.+(1+\epsilon)\int_{0}^{1-\epsilon}(1-\epsilon)^{-\epsilon}d(1-\epsilon)\right)$$
$$\lesssim \sup_{0<1+\epsilon<(1-\epsilon)^{\frac{1}{1+\epsilon}}}\frac{\varphi(1+\epsilon)}{1+\epsilon}\left((1+\epsilon)(1+\epsilon)^{(1-\epsilon^2)}+(1+\epsilon)(1-\epsilon)^{1-\epsilon}\right)$$
$$\lesssim \varphi\left((1-\epsilon)^{\frac{1}{1+\epsilon}}\right)(1-\epsilon)^{1+\epsilon}$$

$$\leq (1-\epsilon) \sup_{\frac{1}{2}>\epsilon>0} \varphi\left((1+\epsilon)^{\frac{1}{1+\epsilon}}\right)(1+\epsilon)^{-\epsilon},$$

$$\sup_{(1-\epsilon)^{\frac{1}{1+\epsilon}}<1+\epsilon<(-\epsilon)^{\frac{1}{1+\epsilon}}-0<(1+\epsilon)^{1+\epsilon}-1+\epsilon} V(1+\epsilon,1-\epsilon,1-\epsilon)$$

$$\leq \sup_{(1-\epsilon)^{\frac{1}{1+\epsilon}}<1+\epsilon<(-\epsilon)^{\frac{1}{1+\epsilon}}} \frac{\varphi(1+\epsilon)}{1+\epsilon}\left(\int_{(1+\epsilon)^{1+\epsilon}-(1-\epsilon)}^{(1+\epsilon)^{1+\epsilon}} (1-\epsilon)^{\frac{\epsilon^2+\epsilon-1}{1+\epsilon}} d(1-\epsilon)\right.$$

$$\left. + (1+\epsilon)\int_{(1+\epsilon)^{1+\epsilon}}^{(1+\epsilon)^{(1+\epsilon)}+(1-\epsilon)} (1-\epsilon)^{-\epsilon} d(1-\epsilon)\right)$$

$$\leq \sup_{(1-\epsilon)^{\frac{1}{1+\epsilon}}<1+\epsilon<(-\epsilon)^{\frac{1}{1+\epsilon}}} \frac{\varphi(1+\epsilon)}{1+\epsilon}\left((1-\epsilon)(1+\epsilon)^{-\epsilon^2-\epsilon+1} + (1-\epsilon^2)(1+\epsilon)^{-\epsilon(1+\epsilon)}\right)$$

$$\lesssim (1-\epsilon) \sup_{(1-\epsilon)^{\frac{1}{1+\epsilon}}<1-\epsilon<1} \varphi(1+\epsilon)(1+\epsilon)^{-\epsilon}$$

$$\lesssim (1-\epsilon) \sup_{\frac{1}{2}>\epsilon>0} \varphi\left((1+\epsilon)^{\frac{1}{1+\epsilon}}\right)(1+\epsilon)^{-\epsilon}$$

and

$$\sup_{(-\epsilon)^{\frac{1}{1+\epsilon}}<1+\epsilon<1(1+\epsilon)^{1+\epsilon}-0<1} V(1+\epsilon,1-\epsilon,1-\epsilon)$$

$$\leq \sup_{(-\epsilon)^{\frac{1}{1+\epsilon}}<1+\epsilon<1} \frac{\varphi(1+\epsilon)}{1+\epsilon}\left(\int_{(1+\epsilon)^{1+\epsilon}-1-\epsilon}^{(1+\epsilon)^{1+\epsilon}} (1-\epsilon)^{\frac{\epsilon^2+\epsilon-1}{1+\epsilon}} d(1-\epsilon)\right.$$

$$\left. + (1+\epsilon)\int_{(1+\epsilon)^{1+\epsilon}}^{1} (1-\epsilon)^{-\epsilon} d(1-\epsilon)\right)$$

$$\leq \sup_{(-\epsilon)^{\frac{1}{1+\epsilon}}<1+\epsilon<1} \frac{\varphi(1+\epsilon)}{1+\epsilon}\left((1-\epsilon)(1+\epsilon)^{-\epsilon^2-\epsilon+1}\right.$$

$$\left. + (1+\epsilon)(1-(1+\epsilon)^{1+\epsilon})(1+\epsilon)^{-\epsilon(1+\epsilon)}\right)$$

$$\leq \sup_{(-\epsilon)^{\frac{1}{1+\epsilon}}<1+\epsilon<1} \frac{\varphi(1+\epsilon)}{1+\epsilon}\left((1-\epsilon)(1+\epsilon)^{-\epsilon^2-\epsilon+1}+(1-\epsilon)(1+\epsilon)^{-\epsilon^2-\epsilon+1}\right)$$

$$\lesssim (1-\epsilon) \sup_{(-\epsilon)^{\frac{1}{1+\epsilon}}<1+\epsilon<1} \varphi(1+\epsilon)(1+\epsilon)^{-\epsilon(1+\epsilon)} \lesssim (1-\epsilon) \sup_{\frac{1}{2}>\epsilon>0} \varphi\left((1+\epsilon)^{\frac{1}{1+\epsilon}}\right)(1+\epsilon)^{-\epsilon}.$$

Finally

$$\varphi_X(1-\epsilon) \lesssim (1-\epsilon) \sup_{\frac{1}{2}>\epsilon>0} \varphi(1+\epsilon)(1+\epsilon)^{-\epsilon},\ 0<\epsilon<\frac{1}{2}.$$

$$\varphi_X(1-\epsilon) \lesssim (1-\epsilon) \sup_{\frac{1}{2}>\epsilon>0} \varphi(1+\epsilon)(1+\epsilon)^{-\epsilon},\ 0<\epsilon<\frac{1}{2}.$$

**Proof of Theorem B**. Consider the Orlicz space $L^A(0,1)L^A(0,1)$ and the Marcinkiewicz space $M(0,1)M(0,1)$ from the assumption of the theorem.We will prove both implications at one using only equivalent steps.The statement $H_{1-\epsilon}^{1+\epsilon}: L^A(0,1) \to M(0,1)$ $H_{1-\epsilon}^{1+\epsilon}: L^A(0,1) \to M(0,1)$ means

$$\left\|\int_{(1-\epsilon)^{1+\epsilon}}^{1} \sum g_j(1+\epsilon)(1-\epsilon)^{-\epsilon} d(1+\epsilon)\right\|_{M(0,1)} \lesssim \sum_j \|g_j\|_{L^A(0,1)},\ g_j \in M^+(0,1).$$

Passing to the associate spaces, this by [ 10,Lemma 8.1] the same as

$$\left\|(1-\epsilon)^{-\epsilon} \int_0^{(1-\epsilon)^{\frac{1}{1+\epsilon}}} \sum f_j(1+\epsilon) d(1+\epsilon)\right\|_{L^{\tilde{A}}(0,1)} \lesssim \sum_j \|f_j\|_{M'(0,1)},\ f_j \in M^+(0,1),$$

where$\tilde{A}\tilde{A}$ is the complementary Young function to $AA$.This is equivalent to

$$\left\|(1-\epsilon)^{-\epsilon} \int_0^{(1-\epsilon)^{\frac{1}{1+\epsilon}}} \sum f_j^*(1+\epsilon) d(1+\epsilon)\right\|_{L^{\tilde{A}}(0,1)} \lesssim \sum_j \|f_j\|_{M'(0,1)},\ f_j \in M^+(0,1).$$

Indeed, one implication follows just by passing to only nonincreasing functions with the fact that $\|\Sigma f_j\|_{M'(0,1)} = \|\Sigma f_j^*\|_{M'(0,1)}$, $\|\Sigma f_j\|_{M'(0,1)} = \|\Sigma f_j^*\|_{M'(0,1)}$, and the other holds thanks to the Hardy-Littewood inequality applied to functions $f_j f_j$ and

$\chi_{\left(0,(1-\epsilon)^{\frac{1}{1+\epsilon}}\right)} \cdot \chi_{\left(0,(1-\epsilon)^{\frac{1}{1+\epsilon}}\right)}$.

Using the fact that $M'(0,1)M'(0,1)$is a Lorentz endpoint space and passing to the characteristic functions while keeping Proposition 3.4 in mind, this is equivalent to

$$\left\|(1-\epsilon)^{-\epsilon}\int_0^{(1-\epsilon)^{\frac{1}{1+\epsilon}}} \chi_{(0,a)}(1+\epsilon)d(1+\epsilon)\right\|_{L^{\tilde{A}}(0,1)} \lesssim \varphi M'(a) \quad a \in (0,1). \quad (3.8)$$

Let us compute the left hand side. Clearly

$$\left\|(1-\epsilon)^{-\epsilon}\int_0^{(1-\epsilon)^{\frac{1}{1+\epsilon}}} \chi_{(0,a)}(1+\epsilon)d(1+\epsilon)\right\|_{L^{\tilde{A}}(0,1)}$$

$$= \left\|(1-\epsilon)^{-\epsilon}\chi_{(0,a^{1+\epsilon})}(1-\epsilon)\cdot(1-\epsilon)^{\frac{1}{1+\epsilon}} + (1-\epsilon)^{-\epsilon}\chi_{(a^{1+\epsilon},1)}(1-\epsilon) \cdot a\right\|_{L^{\tilde{A}}(0,1)}$$

$$\leq \left\|(1-\epsilon)^{\frac{\epsilon^2+\epsilon-1}{1+\epsilon}}\chi_{(0,a^{1+\epsilon})}(1-\epsilon)\right\|_{L^{\tilde{A}}(0,1)} + a\left\|(1-\epsilon)^{-\epsilon}\chi_{(a^{1+\epsilon},1)}(1-\epsilon)\right\|_{L^{\tilde{A}}(0,1)}.$$

We suppose that $a \in \left(0, 2^{\frac{-1}{1+\epsilon}}\right)$,$a \in \left(0, 2^{\frac{-1}{1+\epsilon}}\right)$, since we are interested only in values of a near zero. We show that the second summand domainates the first one. Indeed, for any r.i. norm we have

$$a\left\|(1-\epsilon)^{-\epsilon}\chi_{(a^{1+\epsilon},1)}(1-\epsilon)\right\| \geq a\left\|(1-\epsilon)^{-\epsilon}\chi_{(a^{1+\epsilon},2a^{1+\epsilon})}(1-\epsilon)\right\|$$
$$\geq a(2a^{1+\epsilon})^{-\epsilon}\left\|\chi_{(a^{1+\epsilon},2a^{1+\epsilon})}(1-\epsilon)\right\| \simeq a^{-\epsilon^2-\epsilon+1}\left\|\chi_{(0,a^{1+\epsilon})}(1-\epsilon)\right\|$$
$$\geq \left\|(1-\epsilon)^{\frac{\epsilon^2+\epsilon-1}{1+\epsilon}}\chi_{(0,a^{1+\epsilon})}(1-\epsilon)\right\|.$$

Therefore we can state that

$$\left\|(1-\epsilon)^{-\epsilon}\int_0^{(1-\epsilon)^{\frac{1}{1+\epsilon}}}\chi_{(0,a)}(1+\epsilon)d(1+\epsilon)\right\|_{L^{\tilde{A}}(0,1)}$$
$$\simeq a\left\|(1-\epsilon)^{-\epsilon}\chi_{(a^{1+\epsilon},1)}(1-\epsilon)\right\|_{L^{\tilde{A}}(0,1)}. (3.9)$$

At this moment, it is the time for using Lemma 3.1.We need the part (3.4) with (3.3) for $\xi = \epsilon > 0, R = 1\xi = \epsilon > 0, R = 1$ and $\varepsilon = 1 - 2^{\frac{-1}{1+\epsilon}}.\varepsilon = 1 - 2^{\frac{-1}{1+\epsilon}}$. The assumption (3.1) can be rendered as satisfied without any loss of generality since the domain is of finite measure, hence the appropriate Young function can be redefined on $(0,1)(0,1)$ without any effect to the corresponding Orlicz space.Note also that we are using the complementary Young function$\tilde{A}\tilde{A}$ instead of $AA$. Hence we conclude that (3.8) is equivalent to

$$\frac{a^{-\epsilon^2-\epsilon+1}}{E_{-\epsilon}^{-1}(a^{-(1+\epsilon)})} \lesssim \varphi_{M'}(a) \quad a \in \left(0, 2^{-\frac{1}{1+\epsilon}}\right). \qquad (3.10)$$

Now we substitute $(1-\epsilon) = a^{-(1+\epsilon)}(1-\epsilon) = a^{-(1+\epsilon)}$ and use the fact that $\varphi_{M'}(a)\varphi(a) = a.\varphi_{M'}(a)\varphi(a) = a$. We get

$$\varphi\left((1-\epsilon)^{-\frac{1}{1+\epsilon}}\right)(1-\epsilon)^{-\epsilon} \lesssim E_{-\epsilon}^{-1}(1-\epsilon), -\infty < \epsilon < 2^{-\frac{1}{1+\epsilon}} - 1. \qquad (3.11)$$

$$\varphi\left((1-\epsilon)^{-\frac{1}{1+\epsilon}}\right)(1-\epsilon)^{-\epsilon} \lesssim E_{-\epsilon}^{-1}(1-\epsilon), -\infty < \epsilon < 2^{-\frac{1}{1+\epsilon}} - 1. \qquad (3.11),$$

Letusdefine$F(1-\epsilon)=\overline{\varphi}\left((1-\epsilon)^{-\frac{1}{1+\epsilon}}\right)(1-\epsilon)^{-\epsilon}, \quad 0<\epsilon<1,$

$F(1-\epsilon)=\overline{\varphi}\left((1-\epsilon)^{-\frac{1}{1+\epsilon}}\right)(1-\epsilon)^{-\epsilon}, \quad 0<\epsilon<1,$

where the function $\overline{\varphi}(1-\epsilon)\overline{\varphi}(1-\epsilon)$ is taken from Lemma 3.2. We claim that $F(1-\epsilon)\,F(1-\epsilon)$ is the least nondecreasing majoran of $\varphi\left((1-\epsilon)^{\frac{-1}{1+\epsilon}}\right)(1-\epsilon)^{-\epsilon}\varphi\left((1-\epsilon)^{\frac{-1}{1+\epsilon}}\right)(1-\epsilon)^{-\epsilon}$. Indeed,

$\overline{\varphi}(1-\epsilon)=(1-\epsilon)^{-\epsilon(1+\epsilon)}\sup_{\frac{1}{2}>\epsilon>0}\varphi(1+\epsilon)(1+\epsilon)^{-\epsilon(1+\epsilon)}, 0<\epsilon<1,$

$\overline{\varphi}(1-\epsilon)=(1-\epsilon)^{-\epsilon(1+\epsilon)}\sup_{\frac{1}{2}>\epsilon>0}\varphi(1+\epsilon)(1+\epsilon)^{-\epsilon(1+\epsilon)}, 0<\epsilon<1,$

hence

$\overline{\varphi}\left((1-\epsilon)^{-\frac{1}{1+\epsilon}}\right)(1-\epsilon)^{-\epsilon}=\sup_{0<\epsilon<\frac{1}{2}}\varphi\left((1+\epsilon)^{-\frac{1}{1+\epsilon}}\right)(1+\epsilon)^{-\epsilon}, 0<\epsilon<1,$

$\overline{\varphi}\left((1-\epsilon)^{-\frac{1}{1+\epsilon}}\right)(1-\epsilon)^{-\epsilon}=\sup_{0<\epsilon<\frac{1}{2}}\varphi\left((1+\epsilon)^{-\frac{1}{1+\epsilon}}\right)(1+\epsilon)^{-\epsilon}, 0<\epsilon<1,$

and the claim follows. Since the function $E_{-\epsilon}E_{-\epsilon}$ is strictly increasing as well as its inverse, we can enlarge the left hand side of the inequality (3.11) by $F(1-\epsilon)\,F(1-\epsilon)$. Hence we can equivalently continue by

$$F(1+\epsilon)\lesssim E_{-\epsilon}^{-1}(1-\epsilon), \quad -\infty<\epsilon<2^{-\frac{1}{1+\epsilon}}-1. \tag{3.12}$$

Now Lemma 3.3comes to play with

$u(1-\epsilon)=\overline{\varphi}\left((1-\epsilon)^{\frac{1}{1+\epsilon}}\right)(1-\epsilon)^{1+\epsilon}.$

$u(1-\epsilon)=\overline{\varphi}\left((1-\epsilon)^{\frac{1}{1+\epsilon}}\right)(1-\epsilon)^{1+\epsilon}$. By Lemma 3.2 $u\,u$ is quasiconcave and strictly increasing on $(0,1).(0,1)$. Next,

$$\lim_{1-\epsilon\to 0^+}\frac{u(1-\epsilon)}{1-\epsilon}=\lim_{1-\epsilon\to 0^+}\sup_{\frac{1}{2}>\epsilon>0}\varphi\left((1+\epsilon)^{\frac{1}{1+\epsilon}}\right)(1+\epsilon)^{-\epsilon}=\infty$$

thanks to the assumption of the theorem. Also, $uu$ is right continuous at the origin since

$$u(1-\epsilon)=(1-\epsilon)\sup_{\frac{1}{2}>\epsilon>0}\varphi\left((1+\epsilon)^{\frac{1}{1+\epsilon}}\right)(1+\epsilon)^{-\epsilon}\leq\varphi(1)(1-\epsilon)^{1-\epsilon}.$$

$$u(1-\epsilon)=(1-\epsilon)\sup_{\frac{1}{2}>\epsilon>0}\varphi\left((1+\epsilon)^{\frac{1}{1+\epsilon}}\right)(1+\epsilon)^{-\epsilon}\leq\varphi(1)(1-\epsilon)^{1-\epsilon}.$$

We obtain a Young function $BB$ such that $\tilde{B}^{-1}(1-\epsilon)\simeq F(1-\epsilon)$. $\tilde{B}^{-1}(1-\epsilon)\simeq F(1-\epsilon)$. Theorem C ensures that the space $L^B(0,1)L^B(0,1)$ has the same fundamental function as the optimal r.i. domain $X(0,1)\,X(0,1)$ in $H_{1-\epsilon}^{1+\epsilon}: L^A(0,1)\to M(0,1)$ $H_{1-\epsilon}^{1+\epsilon}: L^A(0,1)\to M(0,1)$. Using this and passing to inverse functions, (3.12) is equivalent to existence of some constant $C>0$ $C>0$ such that

$$E_{-\epsilon}(2+\epsilon)\leq\tilde{B}\big(C(2+\epsilon)\big),0<\epsilon<\infty,$$

$$E_{-\epsilon}(2+\epsilon)\leq\tilde{B}\big(C(2+\epsilon)\big),0<\epsilon<\infty,$$ w h e r e

$c=E_{-\epsilon}^{-1}\left(2^{\frac{1}{1+\epsilon}}\right)>0.c=E_{-\epsilon}^{-1}\left(2^{\frac{1}{1+\epsilon}}\right)>0.$

This is however equivalent to

$$E_{-\epsilon}(2+\epsilon)\lesssim\tilde{B}\big(C(2+\epsilon)\big),\quad 0<\epsilon<\infty,$$

$$E_{-\epsilon}(2+\epsilon)\lesssim\tilde{B}\big(C(2+\epsilon)\big),\quad 0<\epsilon<\infty,$$ w h i c h

is nothing but $\int_0^{2+\epsilon}\frac{\tilde{A}(1-\epsilon)}{(1-\epsilon)^{\frac{1}{1-\epsilon}}}\,d(1-\epsilon)\lesssim\frac{\tilde{B}\big(C(2+\epsilon)\big)}{(2+\epsilon)^{\frac{1}{-\epsilon}}}$

$$\int_0^{2+\epsilon}\frac{\tilde{A}(1-\epsilon)}{(1-\epsilon)^{\frac{1}{1-\epsilon}}}\,d(1-\epsilon)\lesssim\frac{\tilde{B}\big(C(2+\epsilon)\big)}{(2+\epsilon)^{\frac{1}{-\epsilon}}},\qquad 0<\epsilon<\infty.\,0<\epsilon<\infty.$$

Finally observe that the quantities $\int_0^{2+\epsilon} \tilde{A}(1-\epsilon)(1-\epsilon)^{\frac{1+\epsilon}{-\epsilon}} d(1-\epsilon)$ $\int_0^{2+\epsilon} \tilde{A}(1-\epsilon)(1-\epsilon)^{\frac{1+\epsilon}{-\epsilon}} d(1-\epsilon)$ and $\int_1^{2+\epsilon} \tilde{A}(1-\epsilon)(1-\epsilon)^{\frac{1+\epsilon}{-\epsilon}} d(1-\epsilon)$ $\int_1^{2+\epsilon} \tilde{A}(1-\epsilon)(1-\epsilon)^{\frac{1+\epsilon}{-\epsilon}} d(1-\epsilon)$ are comparable since ,$0 < \epsilon < \infty$.,$0 < \epsilon < \infty$. One can now immediately observe that the resulting inequality does not depend on the behavior of the Young function $\tilde{A}\tilde{A}$ on the interval $(0,1)$. $(0,1)$.

Proof of Theorem A. Before proving Theorem A we need several auxiliary results.

The next theorem is the crucial ingredient in the proof of the main result and it reveals the constructive approach to the nonextistence of an optimal Orlicz domain space in appropriate situations (see [20]).

**Theorem 4.1.** Let Young functions $A\,A$ and $BB$ satisfy for $0 \leq \epsilon < 1 0 \leq \epsilon < 1$ and some $C > 0 C > 0$ the inequality

$$\int_1^{2+\epsilon} \frac{A(1-\epsilon)}{(1-\epsilon)^{\frac{1}{1-\epsilon}}} d(1-\epsilon) \lesssim \frac{B\big(C(2+\epsilon)\big)}{(2+\epsilon)^{\frac{-1}{\epsilon}}}, 0 < \epsilon < \infty. \qquad (4.1)$$

If

$$\lim_{2+\epsilon\to\infty} \frac{B(2+\epsilon)}{(2+\epsilon)^{\frac{-1}{\epsilon}}} = \infty \qquad (4.2)$$

and

$$\lim_{2+\epsilon\to\infty} \sup \frac{(2+\epsilon)^{\frac{-1}{\epsilon}}}{B(\epsilon^2+3\epsilon+2)} \int_1^{2+\epsilon} \frac{B(2+\epsilon)}{(1-\epsilon)^{\frac{-1}{\epsilon}}} d(1-\epsilon) = \infty \qquad (4.3)$$

for every $\epsilon \geq 0,\epsilon \geq 0$, then there exists a Young functions $A_1 A_1$ satisfying $A_1 >> AA_1 >> A$ and

$$\int_1^{2+\epsilon} \frac{A_1(1-\epsilon)}{(1-\epsilon)^{\frac{-1}{\epsilon}}} d(1-\epsilon) \lesssim \frac{B\left(c(2+\epsilon)\right)}{(2+\epsilon)^{\frac{-1}{\epsilon}}}$$

$$\int_1^{2+\epsilon} \frac{A_1(1-\epsilon)}{(1-\epsilon)^{\frac{-1}{\epsilon}}} d(1-\epsilon) \lesssim \frac{B\left(c(2+\epsilon)\right)}{(2+\epsilon)^{\frac{-1}{\epsilon}}}, \qquad ,0 < \epsilon < \infty. \ ,0 < \epsilon < \infty.$$

**Proof.** Let $AA$ and $BB$ be the Young functions from the assumptions. First, we establish an upper bound for $AA$. Namely, for $,0 < \epsilon < \infty ,,0 < \epsilon < \infty ,$

.

$$\frac{B\left(2C(1+\epsilon)\right)}{(1+\epsilon)^{\frac{-1}{\epsilon}}}$$

$$\gtrsim \int_1^{2(1+\epsilon)} \frac{A(1-\epsilon)}{(1-\epsilon)^{\frac{1}{1-\epsilon}}} d(1-\epsilon)$$

$$\geq \int_{1+\epsilon}^{2(1+\epsilon)} \frac{A(1-\epsilon)}{(1-\epsilon)^{\frac{1}{1-\epsilon}}} d(1-\epsilon) \geq A(1+\epsilon) \int_{1+\epsilon}^{2(1+\epsilon)} \frac{1}{(1-\epsilon)^{\frac{1}{1-\epsilon}}} d(1-\epsilon)$$

$$\simeq \frac{A(1-\epsilon)}{(1+\epsilon)^{\frac{-1}{\epsilon}}}.$$

Using this, we obtain the existence of $\gamma > 0\gamma > 0$ such that $\gamma B\left(2C(1+\epsilon)\right) > A(1+\epsilon), 0 < \epsilon < \infty .\ (4.4)$

Now we fix this $\gamma\gamma$ and, for every $,0 < \epsilon < \infty ,,0 < \epsilon < \infty ,$ we define the set

$$G_{(1+\epsilon)}=\left\{0<\epsilon<\infty,\frac{A(1+\epsilon)}{1+\epsilon}\geq\gamma\frac{B\big(2C(1+\epsilon)\big)}{1+\epsilon}\right\}.$$

Since $\frac{A(1-\epsilon)}{1-\epsilon}\frac{A(1-\epsilon)}{1-\epsilon}$ is a nondecreasing mapping from $(0,\infty)(0,\infty)$ onto itself, the sets are nonempty for every $,0<\epsilon<\infty.,0<\epsilon<\infty$. Let us define $\tau=\tau_{(1+\epsilon)}=\inf G_{(1+\epsilon)}$. $\tau=\tau_{(1+\epsilon)}=\inf G_{(1+\epsilon)}$. Observe that, for $,0<\epsilon<\infty,$ $,0<\epsilon<\infty,$

and $0<\epsilon<\frac{1}{2},\ \frac{A(1-\epsilon)}{1+\epsilon}\leq\frac{A(1+\epsilon)}{1+\epsilon}<\gamma\frac{B\big(2C(1+\epsilon)\big)}{1+\epsilon}$
$0<\epsilon<\frac{1}{2},\ \frac{A(1-\epsilon)}{1+\epsilon}\leq\frac{A(1+\epsilon)}{1+\epsilon}<\gamma\frac{B\big(2C(1+\epsilon)\big)}{1+\epsilon}$
thanks to the estimate (4.4). Hence $\tau_{(1+\epsilon)}>1+\epsilon\tau_{(1+\epsilon)}>1+\epsilon$

for every $(1+\epsilon).(1+\epsilon)$. Moreover, since $\frac{A(1+\epsilon)}{1+\epsilon}\frac{A(1+\epsilon)}{1+\epsilon}$ is

continuous, we have the equality

$$\frac{A(\tau)}{\tau}=\gamma\frac{B\big(2C(1+\epsilon)\big)}{1+\epsilon},0<\epsilon<\infty, \qquad (4.5)$$

Let $(1+\epsilon)(1+\epsilon)$be a real number such that $\epsilon\geq 0.\epsilon\geq 0$. Then

$$\lim_{1+\epsilon\to\infty}\sup\frac{A(\tau_{1+\epsilon})}{\tau_{1+\epsilon}}\cdot\frac{1+\epsilon}{A\big(2(\epsilon^2+2\epsilon+1)\big)}=\infty. \qquad (4.6)$$

Indeeed, suppose that there exist $\epsilon\geq 0\epsilon\geq 0$ and some $L>0$ $L>0$ such that there is for all ,,

$0<\epsilon<\infty,0<\epsilon<\infty,$ the estimate $\frac{A(\tau)}{\tau}\cdot\frac{1+\epsilon}{A\big(2(\epsilon^2+2\epsilon+1)\big)}<L,$ $\frac{A(\tau)}{\tau}\cdot\frac{1+\epsilon}{A\big(2(\epsilon^2+2\epsilon+1)\big)}<L,$ or equivalently

$$\frac{A\big(2(\epsilon^2+2\epsilon+1)\big)}{1+\epsilon} > L^{-1}\frac{A(\tau)}{\tau}\ . \qquad (4.7)$$

Now for $,\epsilon > 0\ ,\epsilon > 0$ the following holds:

$$\frac{B\big(C(\epsilon^2+2\epsilon+1)\big)}{(2+\epsilon)^{\frac{1}{1-\epsilon}}} \gtrsim$$

$$\int_1^{(\epsilon^2+2\epsilon+1)}\frac{A(1-\epsilon)}{(1-\epsilon)^{\frac{1}{1-\epsilon}}}d(1-\epsilon) \geq \int_{1+\epsilon}^{(\epsilon^2+2\epsilon+1)}\frac{A(1-\epsilon)}{(1-\epsilon)^{\frac{1}{1-\epsilon}}}d(1-\epsilon) \simeq \int_{\frac{1}{2}}^{\frac{2+\epsilon}{2}}\frac{A\big(2(\epsilon^2-1)\big)}{(1-\epsilon)^{\frac{1}{1-\epsilon}}}d(1-\epsilon)$$

$$\frac{B\big(C(\epsilon^2+2\epsilon+1)\big)}{(2+\epsilon)^{\frac{1}{1-\epsilon}}} \gtrsim$$

$$\int_1^{(\epsilon^2+2\epsilon+1)}\frac{A(1-\epsilon)}{(1-\epsilon)^{\frac{1}{1-\epsilon}}}d(1-\epsilon) \geq \int_{1+\epsilon}^{(\epsilon^2+2\epsilon+1)}\frac{A(1-\epsilon)}{(1-\epsilon)^{\frac{1}{1-\epsilon}}}d(1-\epsilon) \simeq \int_{\frac{1}{2}}^{\frac{2+\epsilon}{2}}\frac{A\big(2(\epsilon^2-1)\big)}{(1-\epsilon)^{\frac{1}{1-\epsilon}}}d(1-\epsilon)$$

(by change of variables) $\gtrsim \int_{\frac{1}{2}}^{\frac{2+\epsilon}{2}}\frac{A\big(\tau_{(1-\epsilon)}\big)}{\tau_{(1-\epsilon)}}\frac{1}{(1-\epsilon)^{\frac{1}{1-\epsilon}}}d(1-\epsilon)$

$$\gtrsim \int_{\frac{1}{2}}^{\frac{2+\epsilon}{2}}\frac{A\big(\tau_{(1-\epsilon)}\big)}{\tau_{(1-\epsilon)}}\frac{1}{(1-\epsilon)^{\frac{1}{1-\epsilon}}}d(1-\epsilon)$$

(by (4.7)) $\simeq \int_{\frac{1}{2}}^{\frac{2+\epsilon}{2}}\frac{B\big(2C(1-\epsilon)\big)}{1-\epsilon}\frac{1}{(1-\epsilon)^{\frac{1}{-\epsilon}}}d(1-\epsilon)$

$\simeq \int_{\frac{1}{2}}^{\frac{2+\epsilon}{2}}\frac{B\big(2C(1-\epsilon)\big)}{1-\epsilon}\frac{1}{(1-\epsilon)^{\frac{1}{-\epsilon}}}d(1-\epsilon)$(by (4.5))

$$\simeq \int_1^{2+\epsilon}\frac{B\big(C(1-\epsilon)\big)}{(1-\epsilon)^{\frac{1}{1-\epsilon}}}d(1-\epsilon).\simeq \int_1^{2+\epsilon}\frac{B\big(C(1-\epsilon)\big)}{(1-\epsilon)^{\frac{1}{1-\epsilon}}}d(1-\epsilon).$$

Afterthechangeofvariables $C(1-\epsilon)\mapsto 1-\epsilon, C(1-\epsilon)\mapsto 1-\epsilon,$ this contradicts (4.3) for this $(1+\epsilon)(1+\epsilon)$. From estimate (4.6), we can take an increasing sequence $2<(2+\epsilon)_j<\infty, j\geq 2,$ $2<(2+\epsilon)_j<\infty, j\geq 2,$ such that

$$\lim_{j\to\infty} \frac{A(\tau_j)}{\tau_j} \cdot \frac{(2+\epsilon)_j}{A\left(j(2+\epsilon)_j\right)} = \infty, \qquad (4.8)$$

where we define $\tau_j = \tau_{(2+\epsilon)_j}.\tau_j = \tau_{(2+\epsilon)_j}$. We can also choose this sequence to ensure $\tau_{j+1} > \tau_j$ .$\tau_{j+1} > \tau_j$ .We claim that without loss of generality we can assume that $2\tau_j < \tau_j 2\tau_j < \tau_j$ for every index $j \geq 2.j \geq 2$. Indeed, suppose that there exists a subsequence $j_{(1+\epsilon)} j_{(1+\epsilon)}$ in $\mathbb{N}\mathbb{N}$ such that $\tau_{j_{(1+\epsilon)}} \leq 2\tau_{j_{(1+\epsilon)}}$ .$\tau_{j_{(1+\epsilon)}} \leq 2\tau_{j_{(1+\epsilon)}}$ . Then

$A\left(\tau_{j_{(1+\epsilon)}}\right) \leq A\left(2\tau_{j_{(1+\epsilon)}}\right) A\left(\tau_{j_{(1+\epsilon)}}\right) \leq A\left(2\tau_{j_{(1+\epsilon)}}\right)$ and

$$\frac{A\left(\tau_{j_{(1+\epsilon)}}\right)}{\tau_{j_{(1+\epsilon)}}} \cdot \frac{(2+\epsilon)_{j_{(1+\epsilon)}}}{A\left(j_{(1+\epsilon)}(2+\epsilon)_{j_{(1+\epsilon)}}\right)} \leq \frac{A\left(2(2+\epsilon)_{j_{(1+\epsilon)}}\right)}{(2+\epsilon)_{j_{(1+\epsilon)}}} \cdot \frac{(2+\epsilon)_{j_{(1+\epsilon)}}}{A\left(\frac{j_{1+\epsilon}}{2} 2(2+\epsilon)_{j_{(1+\epsilon)}}\right)} \leq \frac{A\left(2(2+\epsilon)_{j_{(1+\epsilon)}}\right)}{A\left(2(2+\epsilon)_{j_{(1+\epsilon)}}\right)} \cdot \frac{2}{j_{(1+\epsilon)}} = \frac{2}{j_{(1+\epsilon)}} \to 0$$

$$\frac{A\left(\tau_{j_{(1+\epsilon)}}\right)}{\tau_{j_{(1+\epsilon)}}} \cdot \frac{(2+\epsilon)_{j_{(1+\epsilon)}}}{A\left(j_{(1+\epsilon)}(2+\epsilon)_{j_{(1+\epsilon)}}\right)} \leq \frac{A\left(2(2+\epsilon)_{j_{(1+\epsilon)}}\right)}{(2+\epsilon)_{j_{(1+\epsilon)}}} \cdot \frac{(2+\epsilon)_{j_{(1+\epsilon)}}}{A\left(\frac{j_{1+\epsilon}}{2} 2(2+\epsilon)_{j_{(1+\epsilon)}}\right)} \leq \frac{A\left(2(2+\epsilon)_{j_{(1+\epsilon)}}\right)}{A\left(2(2+\epsilon)_{j_{(1+\epsilon)}}\right)} \cdot \frac{2}{j_{(1+\epsilon)}} = \frac{2}{j_{(1+\epsilon)}} \to 0$$

as $1+\epsilon \to \infty$, $1+\epsilon \to \infty$,
which is impossible due to (4.8).
At this moment, we can define a function $A_1 A_1$ by the formula

$$A_1(2+\epsilon) = \begin{cases} A\left((2+\epsilon)_j\right) + \dfrac{A(\tau_j) - A\left((2+\epsilon)_j\right)}{\tau_j - (2+\epsilon)_j}\left(2+\epsilon-(2+\epsilon)_j\right), & (2+\epsilon) \\ A(2+\epsilon), & \text{otherwise.} \end{cases}$$
$$\leq (2+\epsilon)_j \leq \tau_j, j \in \mathbb{N},$$

Obviously, $A_1 \geq A A_1 \geq A$ and $A_1 A_1$ is a Young function.

Moreover, for $j \in \mathbb{N},\ j \geq 2$,$j \in \mathbb{N},\ j \geq 2$,

$$\frac{A_1\left(2(2+\epsilon)_j\right)}{A\left(j(2+\epsilon)_j\right)} = \frac{A\left((2+\epsilon)_j\right) + \frac{A(\tau_j) - A\left((2+\epsilon)_j\right)}{\tau_j - (2+\epsilon)_j}(2+\epsilon)_j}{A\left(j(2+\epsilon)_j\right)} \geq \frac{A(\tau_j) - A\left((2+\epsilon)_j\right)}{A\left(j(2+\epsilon)_j\right)} \cdot \frac{(2+\epsilon)_j}{\tau_j} \geq \frac{A(\tau_j) - A\left(\frac{\tau_j}{2}\right)}{A\left(j(2+\epsilon)_j\right)} \cdot \frac{(2+\epsilon)_j}{\tau_j}$$

$$\frac{A_1\left(2(2+\epsilon)_j\right)}{A\left(j(2+\epsilon)_j\right)} = \frac{A\left((2+\epsilon)_j\right) + \frac{A(\tau_j) - A\left((2+\epsilon)_j\right)}{\tau_j - (2+\epsilon)_j}(2+\epsilon)_j}{A\left(j(2+\epsilon)_j\right)} \geq \frac{A(\tau_j) - A\left((2+\epsilon)_j\right)}{A\left(j(2+\epsilon)_j\right)} \cdot \frac{(2+\epsilon)_j}{\tau_j} \geq \frac{A(\tau_j) - A\left(\frac{\tau_j}{2}\right)}{A\left(j(2+\epsilon)_j\right)} \cdot \frac{(2+\epsilon)_j}{\tau_j}$$

(since $2(2+\epsilon)_j < \tau_j 2(2+\epsilon)_j < \tau_j$) $\geq \frac{1}{2} \cdot \frac{A(\tau_j)}{\tau_j} \cdot \frac{(2+\epsilon)_j}{A\left(j(2+\epsilon)_j\right)}$

$\geq \frac{1}{2} \cdot \frac{A(\tau_j)}{\tau_j} \cdot \frac{(2+\epsilon)_j}{A\left(j(2+\epsilon)_j\right)}$ (since$A\left(\frac{\tau_j}{2}\right) \leq A\left(\frac{\tau_j}{2}\right) A\left(\frac{\tau_j}{2}\right) \leq A\left(\frac{\tau_j}{2}\right)$), and the latter tends to infinity as $j \to \infty j \to \infty$ by (4.8). Therefore

$\lim_{2+\epsilon \to \infty} \sup \frac{A_1(2+\epsilon)}{A(\lambda(2+\epsilon))} = \infty \lim_{2+\epsilon \to \infty} \sup \frac{A_1(2+\epsilon)}{A(\lambda(2+\epsilon))} = \infty$ for every $\lambda > 2$,$\lambda > 2$, which is precisely $A_1 >> A.A_1 >> A$. It remains to show that $A_1 A_1$ satisfies the condition (4.1) with $AA$ replaced by $A_1.A_1$.Let $0 < \epsilon < \infty\, 0 < \epsilon < \infty$ be fixed. We find $j \in \mathbb{N} j \in \mathbb{N}$ such that $(2+\epsilon) \leq (2+\epsilon)_j < (2+\epsilon)_{j+1}$. $(2+\epsilon) \leq (2+\epsilon)_j < (2+\epsilon)_{j+1}$. Then we have

$$\int_1^{2+\epsilon} \frac{A_1(1-\epsilon)}{(1-\epsilon)^{\frac{1}{1-\epsilon}}} d(1-\epsilon)$$

$$\leq \int_1^{2+\epsilon} \frac{A(1-\epsilon)}{(1-\epsilon)^{\frac{1}{1-\epsilon}}} d(1-\epsilon)$$

$$+\sum_{\epsilon=0}^{j} \int_{(2+\epsilon)_{1+\epsilon}}^{\tau_{1+\epsilon}} \Bigg( A((2+\epsilon)_{1+\epsilon})$$

$$+\frac{A(\tau_{1+\epsilon}) - A((2+\epsilon)_{1+\epsilon})}{\tau_{1+\epsilon} - (2+\epsilon)_{1+\epsilon}} \left((1-\epsilon) - (2+\epsilon)_{1+\epsilon}\right) \Bigg) \frac{1}{(1-\epsilon)^{\frac{1}{1-\epsilon}}} d(1-\epsilon)$$

$$\leq 2\int_1^{2+\epsilon} \frac{A(1-\epsilon)}{(1-\epsilon)^{\frac{1}{1-\epsilon}}} d(1-\epsilon)$$

$$+\sum_{\epsilon=0}^{j} \frac{A(\tau_{1+\epsilon}) - A((2+\epsilon)_{1+\epsilon})}{\tau_{1+\epsilon} - (2+\epsilon)_{1+\epsilon}} \int_{(2+\epsilon)_{1+\epsilon}}^{\tau_{1+\epsilon}} \frac{\left((1-\epsilon) - (2+\epsilon)_{1+\epsilon}\right)}{(1-\epsilon)^{\frac{1}{1-\epsilon}}} d(1-\epsilon).$$

We can follow with estimates of the latter integral. Since $\epsilon > -1$, $\epsilon > -1$,

we have for $1+\epsilon \in \mathbb{N} 1+\epsilon \in \mathbb{N}$ such that $0 \leq \epsilon \leq j-1$, $0 \leq \epsilon \leq j-1$,

$$\int_{(2+\epsilon)_{1+\epsilon}}^{\tau_{1+\epsilon}} \frac{\left((1-\epsilon) - (2+\epsilon)_{1+\epsilon}\right)}{(1-\epsilon)^{\frac{1}{1-\epsilon}}} d(1-\epsilon)$$

$$\leq \int_{(2+\epsilon)_{1+\epsilon}}^{\tau_{1+\epsilon}} \frac{1}{(1-\epsilon)^{\frac{-1}{\epsilon}}} d(1-\epsilon) \leq \int_{(2+\epsilon)_{1+\epsilon}}^{\infty} \frac{1}{(1-\epsilon)^{\frac{-1}{\epsilon}}} d(1-\epsilon) \simeq \frac{1}{1+\epsilon}.$$

This together with the fact that $2(2+\epsilon)_{1+\epsilon} < \tau_{1+\epsilon}$ $2(2+\epsilon)_{1+\epsilon} < \tau_{1+\epsilon}$ gives

$$\int_1^{2+\epsilon} \frac{A_1(1-\epsilon)}{(1-\epsilon)^{\frac{1}{1-\epsilon}}} d(1-\epsilon) \lesssim 2\int_1^{2+\epsilon} \frac{A(1-\epsilon)}{(1-\epsilon)^{\frac{1}{1-\epsilon}}} d(1-\epsilon) + 2\sum_{\epsilon=0}^{j} \frac{A(\tau_{1+\epsilon})}{\tau_{1+\epsilon}} \frac{1}{(2+\epsilon)_{1+\epsilon}^{\frac{1}{-1-\epsilon}}}.$$

Since (4.5) implies $\frac{A(\tau_{1+\epsilon})}{\tau_{1+\epsilon}} \frac{1}{(2+\epsilon)_{1+\epsilon}^{\frac{1}{-1-\epsilon}}} = \gamma \frac{B(2C(2+\epsilon)_{1+\epsilon})}{(2+\epsilon)_{1+\epsilon}^{\frac{-1}{\epsilon}}}$ $\frac{A(\tau_{1+\epsilon})}{} \frac{1}{1} = \gamma \frac{B(2C(2+\epsilon)_{1+\epsilon})}{-1}$ we have by (4.1)

$$\int_1^{2+\epsilon} \frac{A_1(1-\epsilon)}{(1-\epsilon)^{\frac{1}{1-\epsilon}}} d(1-\epsilon) \lesssim \frac{B\big(C(2+\epsilon)\big)}{(2+\epsilon)^{\frac{-1}{\epsilon}}} + \sum_{\epsilon=0}^{j} \frac{B(2C(2+\epsilon)_{1+\epsilon})}{(2+\epsilon)_{1+\epsilon}^{\frac{-1}{\epsilon}}}.$$

Because the sequence $(2+\epsilon)_j(2+\epsilon)_j$ could be taken arbitrarily fast growing, we can assume without loss of generality that

$$\frac{B\big(C(2+\epsilon)\big)}{(2+\epsilon)^{\frac{-1}{\epsilon}}} \geq \sum_{\epsilon=0}^{i-1} \frac{B(2C(2+\epsilon)_{1+\epsilon})}{(2+\epsilon)_{1+\epsilon}^{\frac{-1}{\epsilon}}}, \ (2+\epsilon)_j < (2+\epsilon) < \infty,$$

$$\frac{B\big(C(2+\epsilon)\big)}{(2+\epsilon)^{\frac{-1}{\epsilon}}} \geq \sum_{\epsilon=0}^{i-1} \frac{B(2C(2+\epsilon)_{1+\epsilon})}{(2+\epsilon)_{1+\epsilon}^{\frac{-1}{\epsilon}}}, \ (2+\epsilon)_j < (2+\epsilon) < \infty,$$

thanks to the assumption (4.2).

Adding all the estimates together, we finally obtain that

$$\int_1^{2+\epsilon} \frac{A_1(1-\epsilon)}{(1-\epsilon)^{\frac{1}{1-\epsilon}}} d(1-\epsilon) \lesssim \frac{B\big(C(2+\epsilon)\big)}{(2+\epsilon)^{\frac{-1}{\epsilon}}}$$

which proves the theorem. The following auxiliary fact is based on the idea of L'Hôpital's rule and the proof is very similar to the proof of the original result, hence we omit it.

**Proposition 4.2.**Suppose that $f_j f_j$ and $g_j g_j$ are real functions having finite derivatives on some neighborhood of infinity. If $g_j(x) \to \infty, g_j(x) \to \infty$, as $x \to \infty, x \to \infty$, then

$$\liminf_{x\to\infty}\sum\frac{f_j'(x)}{f_j'(x)}\le\liminf_{x\to\infty}\sum\frac{f_j(x)}{g_j(x)}.$$

**Theorem 4.3.** Let $G:(0,\infty)\to(0,\infty)G:(0,\infty)\to(0,\infty)$ be a continuous nondecreasing function satisfying $\Delta_2\Delta_2$ condition. Then the following are equivalent.

(i) $\lim_{x\to\infty}\sup\frac{1}{G(\epsilon^2+3\epsilon+2)}\int_1^{2+\epsilon}\frac{G(1-\epsilon)}{1-\epsilon}d(1-\epsilon)=\infty$ $\lim_{x\to\infty}\sup\frac{1}{G(\epsilon^2+3\epsilon+2)}\int_1^{2+\epsilon}\frac{G(1-\epsilon)}{1-\epsilon}d(1-\epsilon)=\infty$ for every $\epsilon\ge 0;$ $\epsilon\ge 0;$

(ii) $\lim_{x\to\infty}\sup\frac{1}{G(2+\epsilon)}\int_1^{2+\epsilon}\frac{G(1-\epsilon)}{1-\epsilon}d(1-\epsilon)=\infty;$ $\lim_{x\to\infty}\sup\frac{1}{G(2+\epsilon)}\int_1^{2+\epsilon}\frac{G(1-\epsilon)}{1-\epsilon}d(1-\epsilon)=\infty;$

(iii) $\lim_{x\to\infty}\inf\frac{G(\epsilon^2+3\epsilon+2)}{G(2+\epsilon)}=1$ $\lim_{x\to\infty}\inf\frac{G(\epsilon^2+3\epsilon+2)}{G(2+\epsilon)}=1$ for every $\epsilon\ge 0.\epsilon\ge 0.$

**Proof.** The equivalence (ii)⟺⟺(i) is trivial, since the quantites $G(2+\epsilon)\,G(2+\epsilon)$ and $G(\epsilon^2+3\epsilon+2)G(\epsilon^2+3\epsilon+2)$ are comparable for every fixed $\epsilon\ge 0\epsilon\ge 0$ thanks to the fact that $G\in\Delta_2 G\in\Delta_2$ .

Let us focus on the implication (iii)⟹⟹(ii). Let $\epsilon\ge 0\epsilon\ge 0$ be fixed and suppose $\epsilon>-1.\epsilon>-1$. Then

$$\int_1^{(\epsilon^2+3\epsilon+2)}G(1-\epsilon)\frac{d(1-\epsilon)}{1-\epsilon}$$

$$\ge\int_{2+\epsilon}^{(\epsilon^2+3\epsilon+2)}G(1-\epsilon)\frac{d(1-\epsilon)}{1-\epsilon}\ge G(2+\epsilon)\int_{2+\epsilon}^{(\epsilon^2+3\epsilon+2)}\frac{d(1-\epsilon)}{1-\epsilon}$$

$$=G(2+\epsilon)\log(1+\epsilon).$$

Dividing both sides by $G(\epsilon^2+3\epsilon+2)G(\epsilon^2+3\epsilon+2)$ with obtain

$$\log(1+\epsilon)\frac{G(2+\epsilon)}{G(\epsilon^2+3\epsilon+2)} \le \frac{1}{G(\epsilon^2+3\epsilon+2)}\int_1^{(\epsilon^2+3\epsilon+2)}\frac{G(1-\epsilon)}{1-\epsilon}d(1-\epsilon).$$

$$\log(1+\epsilon)\frac{G(2+\epsilon)}{G(\epsilon^2+3\epsilon+2)} \le \frac{1}{G(\epsilon^2+3\epsilon+2)}\int_1^{(\epsilon^2+3\epsilon+2)}\frac{G(1-\epsilon)}{1-\epsilon}d(1-\epsilon).$$

Taking the limes superior as $\epsilon\to\infty\epsilon\to\infty$ on both sides of the inequality, we get

$$\log(1+\epsilon) = \log(1+\epsilon)\limsup_{\epsilon\to\infty}\frac{G(2+\epsilon)}{G(\epsilon^2+3\epsilon+2)}$$

$$\le \limsup_{\epsilon\to\infty}\frac{1}{G(\epsilon^2+3\epsilon+2)}\int_1^{(\epsilon^2+3\epsilon+2)}\frac{G(1-\epsilon)}{1-\epsilon}d(1-\epsilon) = L,$$

where $L\,L$ is independent of $(1+\epsilon)(1+\epsilon)$. Since $\log(1+\epsilon).\le L$ $\log(1+\epsilon).\le L$ for arbitrary $(1+\epsilon),L(1+\epsilon),L$ has no other option but to equal infinity.

To prove (ii)⟹⟹(iii), let $\epsilon\ge 0\epsilon\ge 0$ be fixed and let us define $f_j(2+\epsilon)=\int_1^{2+\epsilon}G((\epsilon^2-1))d\frac{d(1-\epsilon)}{1-\epsilon}$ $f_j(2+\epsilon)=\int_1^{2+\epsilon}G((\epsilon^2-1))d\frac{d(1-\epsilon)}{1-\epsilon}$ and $g(2+\epsilon)=\int_1^{2+\epsilon}G(1-\epsilon)\frac{d(1-\epsilon)}{1-\epsilon}.$ $g(2+\epsilon)=\int_1^{2+\epsilon}G(1-\epsilon)\frac{d(1-\epsilon)}{1-\epsilon}.$ Then both $f_j$ $f_j$ and $g_jg_j$ are continuous and have derivatives, namely $f_j'(2+\epsilon)=\frac{G(\epsilon^2+3\epsilon+2)}{2+\epsilon}g_j'(2+\epsilon)=\frac{G(2+\epsilon)}{2+\epsilon}.$ $f_j'(2+\epsilon)=\frac{G(\epsilon^2+3\epsilon+2)}{2+\epsilon}g_j'(2+\epsilon)=\frac{G(2+\epsilon)}{2+\epsilon}.$ Since (ii) holds, it has to be $g_j(2+\epsilon)\epsilon \to \infty, g_j(2+\epsilon)\epsilon \to \infty$, as $\epsilon \to \infty.\epsilon \to \infty$. Using Proposition 4.2,we get

$$0 \le \liminf_{\epsilon\to\infty} \frac{G(\epsilon^2+3\epsilon+2)}{G(2+\epsilon)} - 1 \le \liminf_{\epsilon\to\infty} \frac{\int_1^{2+\epsilon} G(\epsilon^2-1)\frac{d(1-\epsilon)}{1-\epsilon}}{\int_1^{2+\epsilon} G(1-\epsilon)\frac{d(1-\epsilon)}{1-\epsilon}} - 1$$

$$\le \liminf_{\epsilon\to\infty} \frac{\int_{1+\epsilon}^{(\epsilon^2+3\epsilon+2)} G(1-\epsilon)\frac{d(1-\epsilon)}{1-\epsilon} - \int_1^{2+\epsilon} G(1-\epsilon)\frac{d(1-\epsilon)}{1-\epsilon}}{\int_1^{2+\epsilon} G(1-\epsilon)\frac{d(1-\epsilon)}{1-\epsilon}}$$

$$\le \liminf_{\epsilon\to\infty} \frac{G(2+\epsilon)}{\int_1^{2+\epsilon} G(1-\epsilon)\frac{d(1-\epsilon)}{1-\epsilon}} \frac{\int_{2+\epsilon}^{(\epsilon^2+3\epsilon+2)} G(1-\epsilon)\frac{d(1-\epsilon)}{1-\epsilon}}{G(2+\epsilon)}.$$

Since $\liminf_{\epsilon\to\infty} \frac{G(2+\epsilon)}{\int_1^{2+\epsilon} G(1-\epsilon)\frac{d(1-\epsilon)}{1-\epsilon}} = 0$ $\liminf_{\epsilon\to\infty} \frac{G(2+\epsilon)}{\int_1^{2+\epsilon} G(1-\epsilon)\frac{d(1-\epsilon)}{1-\epsilon}} = 0$ it suffices to show that $\frac{1}{G(2+\epsilon)}\int_{2+\epsilon}^{(\epsilon^2+3\epsilon+2)} G(1-\epsilon)\frac{d(1-\epsilon)}{1-\epsilon}$ $\frac{1}{G(2+\epsilon)}\int_{2+\epsilon}^{(\epsilon^2+3\epsilon+2)} G(1-\epsilon)\frac{d(1-\epsilon)}{1-\epsilon}$ is bounded.To this end we use the fact that $GG$ is nondecreasing and, due to $G \in \Delta_2, G \in \Delta_2$, there is some $c > 0$ $c > 0$ such that $G(\epsilon^2+3\epsilon+2) \le cG(2+\epsilon)$ $G(\epsilon^2+3\epsilon+2) \le cG(2+\epsilon)$ for big $(2+\epsilon)(2+\epsilon)$.For such a $(2+\epsilon)(2+\epsilon)$ we have

$$\frac{1}{G(2+\epsilon)}\int_{2+\epsilon}^{(\epsilon^2+3\epsilon+2)} G(1-\epsilon)\frac{d(1-\epsilon)}{1-\epsilon}$$

$$\leq \frac{G(\epsilon^2+3\epsilon+2)}{G(2+\epsilon)}\int_{2+\epsilon}^{(\epsilon^2+3\epsilon+2)} \frac{d(1-\epsilon)}{1-\epsilon} \leq c\log(1+\epsilon).$$

**Proof of Theorem A.** The equivalence of (ii) and (v) follows directly from Theorem B.The condition (v) holds if and only if (iv) holds thanks to the consequence of [15,Theorem A].In order to show (i)⟹⟹(v) assume that (v) is not satisfied,

i.e., $\lim_{\epsilon\to\infty}\sup\frac{1}{G(\epsilon^2+3\epsilon+2)}\int_1^{(2+\epsilon)}\frac{G(1-\epsilon)}{1-\epsilon}d(1-\epsilon)=\infty$ $\lim_{\epsilon\to\infty}\sup\frac{1}{G(\epsilon^2+3\epsilon+2)}\int_1^{(2+\epsilon)}\frac{G(1-\epsilon)}{1-\epsilon}d(1-\epsilon)=\infty$,for some constant $\epsilon>0,\epsilon>0$, where

$$G(2+\epsilon)=\tilde{B}(2+\epsilon)(2+\epsilon)^{\frac{-1}{\epsilon}}.G(2+\epsilon)=\tilde{B}(2+\epsilon)(2+\epsilon)^{\frac{-1}{\epsilon}}.$$

Now for any Orlicz space$L^A(0,1)L^A(0,1)$ satisfying $H_{1-\epsilon}^{1+\epsilon}: L^A(0,1)\to M(0,1)H_{1-\epsilon}^{1+\epsilon}: L^A(0,1)\to M(0,1)$ there exists a constant $C_A C_A$ such that

$$\int_1^{2+\epsilon}\frac{\tilde{A}(1-\epsilon)}{(1-\epsilon)^{\frac{1}{1-\epsilon}}}d(1-\epsilon)\lesssim\frac{\tilde{B}\left(C_A(2+\epsilon)\right)}{(2+\epsilon)^{\frac{-1}{\epsilon}}},0<\epsilon<\infty$$

due to Theorem B. If the function $GG$ is unbounded, then Theorem 4.1 ensures the existence of a Young function $A_1$ $A_1$ such that the space $L^{A_1}(0,1)L^{A_1}(0,1)$ is strictly larger than $L^A(0,1)L^A(0,1)$ and still renders the inequality above true, with possibly different constants. Now again by Theorem B one has $H_{1-\epsilon}^{1+\epsilon}: L^{A_1}(0,1)\to M(0,1)H_{1-\epsilon}^{1+\epsilon}: L^{A_1}(0,1)\to M(0,1)$ and no optimalOrlicz domain exists. This contradicts (i).The case when the function $GG$ is bounded and hence equivalent to a constant function, corresponds to the situation when $M(0,1) = L^{\infty}(0,1)$. $M(0,1) = L^{\infty}(0,1)$. Then no optimal Orlicz domain exists thanks to a different construction described in [7, Theorem 6.4].This also contradicts (i).

To prove (iii)⟹⟹(i) we claim that$L^{B}(0,1)L^{B}(0,1)$ is among the Orlicz spaces $L^{A}(0,1)L^{A}(0,1)$ the largest space rendering $H^{1+\epsilon}_{1-\epsilon}: L^{A}(0,1) \rightarrow M(0,1).H^{1+\epsilon}_{1-\epsilon}: L^{A}(0,1) \rightarrow M(0,1)$.Indeed let $L^{A}(0,1)$ $L^{A}(0,1)$ be any of such spaces. By the optimality of $X(0,1)$, $X(0,1)$,we have $L^{A}(0,1) \subseteq X(0,1)L^{A}(0,1) \subseteq X(0,1)$ and thus we have the inequality between appropriate fundamental functions $\varphi_X(1-\epsilon) \lesssim \varphi_{L^A},\ \ 0 < \epsilon < 1\ \varphi_X(1-\epsilon) \lesssim \varphi_{L^A},\ \ 0 < \epsilon < 1$.

Since the space $L^{B}(0,1)L^{B}(0,1)$ is defined in a way that its fundamental functions coincides with $\varphi_X\varphi_X$, one gets that $\varphi_{L^B} \lesssim \varphi_{L^A}(1-\epsilon)\varphi_{L^B} \lesssim \varphi_{L^A}(1-\epsilon)$ which implies $A(1-\epsilon) \leq B\big(C(1-\epsilon)\big)A(1-\epsilon) \leq B\big(C(1-\epsilon)\big)$ for some $C > 0, C > 0$, hence $L^{A}(0,1) \subseteq L^{B}(0,1)L^{A}(0,1) \subseteq L^{B}(0,1)$ and $L^{B}(0,1)L^{B}(0,1)$ is optimal. The equivalence of (ii) and (iii) follows directly from the definition of the optimal r.i. space, and the equivalence of (v) and (vi) has already been proved in Theorem 4.3.

**Remark 4.4.**Note that the proof of the implication (iii)⟹⟹(i) does not depend on the target space, so it can be used to prove the optimality in positive cases for any r.i. target space Y.

## 5.Examples and applications

### 5.1. Sobolev embeddings on John domains

We begin by the easiest case of Sobolev embeddings, namely those acting on John domains. We will use the reduction theorem from [10]. Recall that a bounded open set $\Omega\Omega$ in $\mathbb{R}^{(1+2\epsilon)}\mathbb{R}^{(1+2\epsilon)}$ is called a John domain if there exist a constant $c \in (0,1) c \in (0,1)$ and a point $x_0 \in \Omega x_0 \in \Omega$ such that for every $x \in \Omega x \in \Omega$ there exists a rectifiable curve $\varpi: [0,l] \longrightarrow \Omega, \varpi: [0,l] \longrightarrow \Omega$, parameterized by arc length, such that $\varpi(0) = x, \varphi(l) = x_0 \varpi(0) = x, \varphi(l) = x_0$ and $\operatorname{dist}(\varpi(r), \partial\Omega) \leq cr, r \in [0,l] \operatorname{dist}(\varpi(r), \partial\Omega) \leq cr, r \in [0,l]$.
We will use the reduction principle for John domains proved in [10, Theorem 6.1]. It can be read as follows. Let $\epsilon \in \mathbb{N}, \epsilon \geq 0$ $\epsilon \in \mathbb{N}, \epsilon \geq 0$ and let $m \in \mathbb{N}. m \in \mathbb{N}$. Assume that $\Omega\Omega$ is a John domain in $\mathbb{R}^{(2+\epsilon)}.\mathbb{R}^{(2+\epsilon)}$. Let $\|.\|_{X(0,1)}\|.\|_{X(0,1)}$ and $\|.\|_{Y(0,1)}$ $\|.\|_{Y(0,1)}$be rearrangement-invariant function norms. Then the following assertions are equivalent.

(i) The Hardy type inequality $\left\|\sum H_{\frac{m}{2+\epsilon}}^{1} f_j\right\|_{Y(0,1)} \leq C \sum_j \|f_j\|_{X(0,1)}$ $\left\|\sum H_{\frac{m}{2+\epsilon}}^{1} f_j\right\|_{Y(0,1)} \leq C \sum_j \|f_j\|_{X(0,1)}$ holds for some constant $CC$ and for every nonnegative $f_j \in X(0,1) f_j \in X(0,1)$.

(ii) The Sobolev embedding $W^m X(\Omega) \hookrightarrow Y(\Omega)$ $W^m X(\Omega) \hookrightarrow Y(\Omega)$ holds.

Recall that

$W^m X(\Omega) =$
$\{u \in M(\Omega), u \text{ is } m - \text{times weakly differentiable in} \Omega \text{ and} |\nabla^{(1+\epsilon)} u| \in$
$X(\Omega), \epsilon = 0,1,\dots,\mathrm{m}\}$

$W^m X(\Omega) =$
$\{u \in \mathcal{M}(\Omega), u \text{ is } m - \text{times weakly differentiable in} \Omega \text{ and} |\nabla^{(1+\epsilon)} u| \in$
$X(\Omega), \epsilon = 0,1, \dots, m\}$

.

Here, $\nabla^{(1+\epsilon)} u \nabla^{(1+\epsilon)} u$ denotes the vector of all $(1+\epsilon)(1+\epsilon)$-th order weak derivatives of $uu$ and $\nabla^0 u = u.\nabla^0 u = u.$ The norm is then given by $\|u\|_{W^m X(\Omega)} := \sum_{\epsilon=-1}^{m} \|\nabla^{(1+\epsilon)} u\|_{X(\Omega)}$ $\|u\|_{W^m X(\Omega)} := \sum_{\epsilon=-1}^{m} \|\nabla^{(1+\epsilon)} u\|_{X(\Omega)}$. Now, one can select any r.i. space $X(0,1) X(0,1)$ and seek to find an optimal range space. Let $\Omega\Omega$ be a John domain in $\mathbb{R}^{(2+\epsilon)}, m \in \mathbb{N}\, \mathbb{R}^{(2+\epsilon)}, m \in \mathbb{N}$ such that $m < 2+\epsilon m < 2+\epsilon$ and consider the spaces $L^{1+\epsilon} \log^{1+\epsilon} L(\Omega) L^{1+\epsilon} \log^{1+\epsilon} L(\Omega)$ or $L^{1+\epsilon} \log^{1+\epsilon} \log L(\Omega), \epsilon > 0$ $L^{1+\epsilon} \log^{1+\epsilon} \log L(\Omega), \epsilon > 0$ and $1+\epsilon \in \mathbb{R} 1+\epsilon \in \mathbb{R}$ or $\epsilon = 0$ $\epsilon = 0$ and $, \epsilon \geq -1., \epsilon \geq -1.$ By [10, Theorem 6.12 and Example 6.14] (see also [3, Example 1 and 2]), we have

$$W^m L^{1+\epsilon} \log^{1+\epsilon} L(\Omega) \hookrightarrow \begin{cases} L^{\frac{\epsilon^2+3\epsilon+2}{2+\epsilon-m(1+\epsilon)}} \log^{\frac{\epsilon^2+3\epsilon+2}{2+\epsilon-m(1+\epsilon)}} L(\Omega), & 0 \leq \epsilon < 2\epsilon, \\ \exp L^{\frac{n}{2+\epsilon-m(2+\epsilon)}} L(\Omega), & \epsilon = 0, 1 < \epsilon, \\ \exp\exp L^{\frac{2+\epsilon}{2+\epsilon-m}}(\Omega), \quad \epsilon = 0, & \epsilon = 1, \\ L^\infty(\Omega), \quad \epsilon > 0 \text{ or } \epsilon = 0, & \epsilon = 1, \end{cases}$$

and

$$W^m L^{1+\epsilon}\log^{1+\epsilon}\log L(\Omega) \hookrightarrow \begin{cases} L^{\frac{\epsilon^2+3\epsilon+2}{2+\epsilon-m(1+\epsilon)}}\log^{\frac{\epsilon^2+3\epsilon+2}{2+\epsilon-m(1+\epsilon)}}\log L(\Omega), 0 \le \epsilon < 2\epsilon \ , \\ \exp\left(L^{\frac{2+\epsilon}{2+\epsilon-m}}\log^{\frac{m(1+\epsilon)}{2+\epsilon-m}} L\right)(\Omega)\,, \quad \epsilon = 0\ , \\ L^{\infty}(\Omega), \qquad 0 > \epsilon, \end{cases}$$

and all the targets are optimal among all Orlicz spaces. Let us investigate the optimal Orlicz domains.

**Example 5.1.** **a)** Case

$Y(\Omega) = L^{\frac{\epsilon^2+3\epsilon+2}{2+\epsilon-m(1+\epsilon)}}\log^{\frac{\epsilon^2+3\epsilon+2}{2+\epsilon-m(1+\epsilon)}} L(\Omega), 0 \le \epsilon < 2\epsilon$

$Y(\Omega) = L^{\frac{\epsilon^2+3\epsilon+2}{2+\epsilon-m(1+\epsilon)}}\log^{\frac{\epsilon^2+3\epsilon+2}{2+\epsilon-m(1+\epsilon)}} L(\Omega), 0 \le \epsilon < 2\epsilon$ .The space $Y(\Omega)Y(\Omega)$ is not a Marcinkiewicz space, but instead of $Y(\Omega)$ $Y(\Omega)$ we can take the endpoint space $M_\varphi(\Omega)M_\varphi(\Omega)$ with the same fundamental function as the spac $Y(\Omega)\,Y(\Omega)$, namely $\varphi(1-\epsilon) = (1-\epsilon)^{\frac{2+\epsilon-m(1+\epsilon)}{\epsilon^2+3\epsilon+2}}\log^{\frac{1+\epsilon}{1+\epsilon}}\left(\frac{2}{1-\epsilon}\right)$ $\varphi(1-\epsilon) = (1-\epsilon)^{\frac{2+\epsilon-m(1+\epsilon)}{\epsilon^2+3\epsilon+2}}\log^{\frac{1+\epsilon}{1+\epsilon}}\left(\frac{2}{1-\epsilon}\right)$, $0 < \epsilon < 1$ $0 < \epsilon < 1$. Now, thanks to reduction principle the problem of Sobolev embedding is equivalent to the boundedness of the operator $H_{\frac{1}{\frac{m}{2+\epsilon}}}$ $H_{\frac{1}{\frac{m}{2+\epsilon}}}$.

By Theorem A, the optimal Orlicz domain space exists if and only if

$H_{\frac{1}{\frac{m}{2+\epsilon}}} : L^B(0,1) \to M_\varphi(0,1) H_{\frac{1}{\frac{m}{2+\epsilon}}} : L^B(0,1) \to M_\varphi(0,1)$ where, after some calculations, $B(1-\epsilon) = (1-\epsilon)^{1+\epsilon}\log^{1+\epsilon}(1-\epsilon)$ $B(1-\epsilon) = (1-\epsilon)^{1+\epsilon}\log^{1+\epsilon}(1-\epsilon)$ for large $(1+\epsilon).(1+\epsilon)$. This is however the same as

$W^m L^{1+\epsilon}\log^{1+\epsilon} L(\Omega) \hookrightarrow M_\varphi(\Omega) W^m L^{1+\epsilon}\log^{1+\epsilon} L(\Omega) \hookrightarrow M_\varphi(\Omega)$

which is satisfied since $W^m L^{1+\epsilon}\log^{1+\epsilon} L(\Omega) \hookrightarrow Y(\Omega) \subseteq M_\varphi(\Omega)$ $W^m L^{1+\epsilon}\log^{1+\epsilon} L(\Omega) \hookrightarrow Y(\Omega) \subseteq M_\varphi(\Omega)$. Hence both domain and rangespacein $W^m L^{1+\epsilon}\log^{1+\epsilon} L(\Omega) \hookrightarrow L^{\frac{\epsilon^2+3\epsilon+2}{2+\epsilon-m(1+\epsilon)}}\log^{\frac{\epsilon^2+3\epsilon+2}{2+\epsilon-m(1+\epsilon)}} L(\Omega)$ $W^m L^{1+\epsilon}\log^{1+\epsilon} L(\Omega) \hookrightarrow L^{\frac{\epsilon^2+3\epsilon+2}{2+\epsilon-m(1+\epsilon)}}\log^{\frac{\epsilon^2+3\epsilon+2}{2+\epsilon-m(1+\epsilon)}} L(\Omega)$ are optimal among Orlicz spaces.

**b)** Case $Y(\Omega) = \exp L^{\frac{2+\epsilon}{2+\epsilon-m(2\epsilon)}} L(\Omega)$ , $\epsilon = 0\,, 1 < \epsilon$. $Y(\Omega) = \exp L^{\frac{2+\epsilon}{2+\epsilon-m(2\epsilon)}} L(\Omega)$ , $\epsilon = 0\,, 1 < \epsilon$.

The Orlicz space $Y(\Omega) Y(\Omega)$ coincides with the Marcinkiewicz endpoint space $M_\varphi(\Omega) M_\varphi(\Omega)$ (cf. (2.1)) where $\varphi(1-\epsilon) = \log^{-2\epsilon^2-3\epsilon+1}\left(\frac{2}{1-\epsilon}\right) \varphi(1-\epsilon) = \log^{-2\epsilon^2-3\epsilon+1}\left(\frac{2}{1-\epsilon}\right)$ , $0 < \epsilon < 1$, $0 < \epsilon < 1$, .Again by reduction principle and Theorem A we compute the Young function B and test the boundedness of $H_{\frac{1}{m}}$ $H_{\frac{1}{m}}$ on the space $L^B(0,1) L^B(0,1)$ or check the condition using the function $G(1-\epsilon) = \tilde{B}(1-\epsilon)(1-\epsilon)^{\frac{n}{(m-2+\epsilon)}}$ $G(1-\epsilon) = \tilde{B}(1-\epsilon)(1-\epsilon)^{\frac{n}{(m-2+\epsilon)}}$. We get

**Table 1**

Application of Theorem A for the operator $H_{\frac{1}{m}}$ $H_{\frac{1}{m}}$ and John domain (see [20]).

| $Y(\Omega)$ | $L^B(\Omega)$ | $G(\Omega)$ |
|---|---|---|
| $L^{\frac{\epsilon^2+3\epsilon+2}{2+\epsilon-m(1+\epsilon)}}\log^{\frac{\epsilon^2+3\epsilon+2}{2+\epsilon-m(1+\epsilon)}}L(\Omega)$, $0\le\epsilon<2\epsilon$ | $L^{1+\epsilon}\log^{1+\epsilon}L(\Omega)$ | $(1+\epsilon)^{\frac{2+\epsilon}{m-2+\epsilon}+\frac{1+\epsilon}{\epsilon}}\log^{\frac{1+\epsilon}{\epsilon}}(1+\epsilon)$ |
| $L^{\frac{\epsilon^2+3\epsilon+2}{2+\epsilon-m(1+\epsilon)}}\log^{\frac{\epsilon^2+3\epsilon+2}{2+\epsilon-m(1+\epsilon)}}\log L(\Omega)$, $0\le\epsilon<2\epsilon$ | $L^{1+\epsilon}\log^{1+\epsilon}\log L(\Omega)$ | $(1+\epsilon)^{\frac{2+\epsilon}{m-2+\epsilon}+\frac{1+\epsilon}{\epsilon}}\log^{\frac{1+\epsilon}{\epsilon}}\log(1+\epsilon)$ |
| $\exp L^{\frac{2+\epsilon}{2+\epsilon-m(2+\epsilon)}}L(\Omega)$ , $\epsilon=0,0<\epsilon$ | $L^{1+\epsilon}\log^{1}L(\Omega)$ | $\log^{\frac{2+\epsilon-m(2+\epsilon)}{2+\epsilon-m}}(1+\epsilon)$ |
| $\exp\exp L^{\frac{2+\epsilon}{2+\epsilon-m}}(\Omega)$, $\epsilon=0,0<\epsilon$ | $L^{1+\epsilon}\log^{-(1+\epsilon)}\log L(\Omega)$ | $\log\log(1+\epsilon)$ |
| $\exp\left(L^{\frac{2+\epsilon}{2+\epsilon-m}}\log^{\frac{m(1+\epsilon)}{2+\epsilon-m}}L\right)(\Omega)$ , $\epsilon=0$ | $L^{1+\epsilon}\log^{\epsilon}L\log^{1+\epsilon}\log L(\Omega)$ | $\log(1+\epsilon)\log^{\frac{m(1+\epsilon)}{m-2+\epsilon}}\log(1+\epsilon)$ |
| $L^{\infty}(\Omega)L^{1+2\epsilon}(\Omega)$ | | $1$ |

$$G(1+\epsilon)=\log^{\frac{2+\epsilon-m(2+\epsilon)}{2+\epsilon-m}}(1+\epsilon),0<\epsilon<\infty$$

$$G(1+\epsilon)=\log^{\frac{2+\epsilon-m(2+\epsilon)}{2+\epsilon-m}}(1+\epsilon),0<\epsilon<\infty.$$

Since $\lim_{1+\epsilon\to\infty}\inf\frac{\log^{\frac{2+\epsilon-m(2+\epsilon)}{2+\epsilon-m}}(c(1+\epsilon))}{\log^{\frac{2+\epsilon-m(2+\epsilon)}{2+\epsilon-m(1+\epsilon)}}}=1,$

$\lim_{1+\epsilon\to\infty}\inf\frac{\log^{\frac{2+\epsilon-m(2+\epsilon)}{2+\epsilon-m}}(c(1+\epsilon))}{\log^{\frac{2+\epsilon-m(2+\epsilon)}{2+\epsilon-m(1+\epsilon)}}}=1,$ for every $C\ge 1C\ge 1$ and $GG$ satisfies the $\Delta_2\Delta_2$ condition we conclude that the space $L^{1+\epsilon}\log^{1+\epsilon}L(\Omega)L^{1+\epsilon}\log^{1+\epsilon}L(\Omega)$ is not the largest Orlicz space rendering $W^mL^{1+\epsilon}\log^{1+\epsilon}L(\Omega)\hookrightarrow\exp L^{\frac{2+\epsilon}{2+\epsilon-m(2+\epsilon)}}(\Omega)$

$W^m L^{1+\epsilon}\log^{1+\epsilon} L(\Omega) \hookrightarrow \exp L^{\frac{2+\epsilon}{2+\epsilon-m(2+\epsilon)}}(\Omega)$ and no such Orlicz space exists. Just to compare, the space $L^B(0,1)L^B(0,1)$ from Theorem A is $L^{1+\epsilon}\log^1 L(0,1)L^{1+\epsilon}\log^1 L(0,1)$ which is too large. These two examples give us the outline how to use our results to investigate the optimal Orlicz domains. Other cases can be done in an analogous way and we just present the results (see, Table 1). Observe that the optimal Orlicz domains exist in subcritical cases, i.e. when $0 \leq \epsilon < 2\epsilon ,0 \leq \epsilon < 2\epsilon$, otherwise every Orlicz domain space can be improved.

## 1.2. Sobolev embeddings on Maz'ya classes

Our next applications are in Sobolev embeddings on wider family of subsets so-called Maz'ya classes. Let $\Omega\Omega$ be a domain in $\mathbb{R}^{(2+\epsilon)}$, $\epsilon \geq 0,\mathbb{R}^{(2+\epsilon)}$, $\epsilon \geq 0$, with a normalized Lebesgue measure, i.e.$|\Omega| = 1.|\Omega| = 1$. Define the perimeter of a measurable $EE$ in $\Omega P(E,\Omega) = \mathcal{H}^{(1+\epsilon)}(\Omega \cap \partial^M E)\Omega P(E,\Omega) = \mathcal{H}^{(1+\epsilon)}(\Omega \cap \partial^M E)$ where $\partial^M E \partial^M E$ denotes the essential boundary of $EE$. The isoperimetric function $I_\Omega: [0,1] \to [0,\infty] I_\Omega: [0,1] \to [0,\infty]$ of $\Omega\Omega$ is then given by

$$I_\Omega(1-\epsilon) = \inf\left\{P(E,\Omega), E \subseteq \Omega, 1-\epsilon \leq |E| \leq \tfrac{1}{2}\right\}, 0 \leq \epsilon \leq \tfrac{1}{2}$$

$$I_\Omega(1-\epsilon) = \inf\left\{P(E,\Omega), E \subseteq \Omega, 1-\epsilon \leq |E| \leq \tfrac{1}{2}\right\}, 0 \leq \epsilon \leq \tfrac{1}{2}$$

and $I_\Omega(1-\epsilon) = I_\Omega(-\epsilon) I_\Omega(1-\epsilon) = I_\Omega(-\epsilon)$ if $\frac{1}{2} < \epsilon \leq 1$. $\frac{1}{2} < \epsilon \leq 1$. Given $(1-\epsilon) \in \left[\frac{1}{(2+\epsilon)}, 1\right](1-\epsilon) \in \left[\frac{1}{(2+\epsilon)}, 1\right]$, we denote by $\mathfrak{z}_{(1-\epsilon)}\mathfrak{z}_{(1-\epsilon)}$ the Maz'ya class of all Euclidean domains $\Omega\Omega$ in $\mathbb{R}^{(2+\epsilon)}\mathbb{R}^{(2+\epsilon)}$ such that $I_\Omega(1-\epsilon)C(1-\epsilon)^{1-\epsilon}$

$I_\Omega(1-\epsilon)C(1-\epsilon)^{1-\epsilon}$ for $0 \le \epsilon \le \frac{1}{2} 0 \le \epsilon \le \frac{1}{2}$ for some positive $C.C$. The reduction theorem in the class $\mathfrak{z}_{(1-\epsilon)}\mathfrak{z}_{(1-\epsilon)}$ [10, Theorem 6.4] takes thefollowing form. Let $2+\epsilon \in \mathbb{N}, \epsilon \ge 0, m \in \mathbb{N}$, $2+\epsilon \in \mathbb{N}, \epsilon \ge 0, m \in \mathbb{N}$, and $(1-\epsilon) \in [\frac{1}{(2+\epsilon)}, 1)$ $(1-\epsilon) \in [\frac{1}{(2+\epsilon)}, 1)$.

Let $\|.\|_{X(0,1)}\|.\|_{X(0,1)}$ and $\|.\|_{Y(0,1)}\|.\|_{Y(0,1)}$ be rearrangement-invariant function norms. Assume that there exists a constant $CC$ such that

$$\left\|\sum H^1_{m(-\epsilon)} f_j\right\|_{Y(0,1)} \le C \sum_j \|f_j\|_{X(0,1)} \qquad (5.1)$$

for every nonnegative $f_j \in X(0,1) f_j \in X(0,1)$. Then the Sobolev embedding

$W^m X(\Omega) \hookrightarrow Y(\Omega)(5.2)$

Holds for every $\Omega \in \mathfrak{z}_{(1-\epsilon)}\Omega \in \mathfrak{z}_{(1-\epsilon)}$.

Conversely, if the Sobolev embedding (5.2) holds for every $\Omega \in \mathfrak{z}_{(1-\epsilon)}\Omega \in \mathfrak{z}_{(1-\epsilon)}$, then the inequality (5.1) holds. Notice the main difference between this ststement and reduction principle for John domains. In the case of John domains the equivalence of Sobolev embedding and boundedness of Hardy type operator holds for every single domain $\Omega\Omega$, while in the Maz'ya classes $\Omega\,\Omega$ has to range among all domains in $\mathcal{J}_{(1-\epsilon)}\mathcal{J}_{(1-\epsilon)}$. Let us mention similar examples for Orlicz spaces. Let $mm$ be an integer and $1-\epsilon \in [\frac{1}{(2+\epsilon)}, 1) 1-\epsilon \in [\frac{1}{(2+\epsilon)}, 1)$ such that $m(-\epsilon) < 1$

$m(-\epsilon) < 1$ and assume, $\epsilon > 0, \epsilon > 0$ and $1 + \epsilon \in \mathbb{R} 1 + \epsilon \in \mathbb{R}$ or $\epsilon = 0$ $\epsilon = 0$ and , $\epsilon \geq -1, \epsilon \geq -1$. By [10, Theorem 6.12 and Example 6.14], we have

$$W^m L^{1+\epsilon} \log^{1+\epsilon} L(\Omega) \hookrightarrow \begin{cases} L^{\frac{\epsilon^2+3\epsilon+2}{2+\epsilon-m(1+\epsilon)}} \log^{\frac{\epsilon^2+3\epsilon+2}{2+\epsilon-m(1+\epsilon)}} L(\Omega), & 0 \leq \epsilon < 2\epsilon, \\ \exp L^{\frac{2+\epsilon}{(2+\epsilon)-m(2+\epsilon)}} L(\Omega), & \epsilon = 0, 1 < \epsilon, \\ \exp\exp L^{\frac{2+\epsilon}{2+\epsilon-m}}(\Omega), & \epsilon = 0, \epsilon = 1, \\ L^\infty(\Omega), \ 0 > \epsilon \text{ or } \epsilon = 0, & \epsilon = 1, \end{cases}$$

and

$$W^m L^{1+\epsilon} \log^{1+\epsilon} \log L(\Omega) \hookrightarrow \begin{cases} L^{\frac{\epsilon^2+3\epsilon+2}{2+\epsilon-m(1+\epsilon)}} \log^{\frac{\epsilon^2+3\epsilon+2}{2+\epsilon-m(1+\epsilon)}} \log L(\Omega), 0 \leq \epsilon < 2\epsilon, \\ \exp\left(L^{\frac{2+\epsilon}{2+\epsilon-m}} \log^{\frac{m(1+\epsilon)}{2+\epsilon-m}} L\right)(\Omega), & \epsilon = 0, \\ L^\infty(\Omega), & 0 > \epsilon, \end{cases}$$

Moreover, the target spaces are optimal among all Orlicz spaces, as $\Omega\Omega$ ranges in $\mathfrak{F}_{(1-\epsilon)} \mathfrak{F}_{(1-\epsilon)}$.

**Table 2**

Application of Theorem A for the operator $H^1_{m(-\epsilon)} H^1_{m(-\epsilon)}$ and Maz'ya class (see [20]).

| $Y(\Omega)$ | $L^B(\Omega)$ | $G(\Omega)$ |
|---|---|---|

$L^{\frac{1+\epsilon}{1-m(-\epsilon-\epsilon^2)}}\log^{\frac{1+\epsilon}{m(-\epsilon-\epsilon^2)}}L(\Omega)$ $L^{1+\epsilon}\log^{1+\epsilon}L(\Omega)$ $(1+\epsilon)^{\frac{1}{m(-\epsilon-1)}+\frac{1+\epsilon}{\epsilon}}\log^{\frac{1+\epsilon}{\epsilon}}(1+\epsilon)$

$$0 \le \epsilon < \frac{1}{m(-\epsilon)} - 1$$

$L^{\frac{1+\epsilon}{1-m(\epsilon+\epsilon^2)}}\log^{\frac{1+\epsilon}{1-m(\epsilon+\epsilon^2)}}\log L(\Omega)$ $L^{1+\epsilon}\log^{1+\epsilon}\log L(\Omega)$ $(1+\epsilon)^{\frac{1}{m(\epsilon-1)}+\frac{1+\epsilon}{\epsilon}}\log^{\frac{1+\epsilon}{\epsilon}}\log(1+\epsilon)$

$$0 \le \epsilon < \frac{1}{m(\epsilon)} - 1$$

$\exp L^{\frac{1}{1-(2+\epsilon)m(\epsilon)}}(\Omega)\epsilon = \frac{1}{m(\epsilon)} - 1$, $L^{1+\epsilon}\log^{1}L(\Omega)$ $\log^{\frac{1-(2+\epsilon)m(\epsilon)}{1-m(\epsilon)}}(1+\epsilon)$

$\exp\exp L^{\frac{1}{1-m(\epsilon)}}(\Omega)$, $\epsilon = \frac{1}{m(\epsilon)} - 1$ $L^{1+\epsilon}\log^{-(1+\epsilon)}\log L(\Omega)$ $\log\log(1+\epsilon)$

$\exp\left(L^{\frac{1}{1-m(\epsilon)}}\log^{\frac{m(\epsilon+\epsilon^2)}{1-m(\epsilon)}}L\right)(\Omega)$ $L^{1+\epsilon}\log^{\epsilon}L\log^{1+\epsilon}\log L(\Omega)\log($ $\log^{\frac{m(\epsilon+\epsilon^2)}{m(\epsilon-1)}}\log(1+\epsilon)$

$$\epsilon = \frac{1}{m(\epsilon)} - 1$$

$L^{\infty}(\Omega)L^{m(\epsilon)}(\Omega)$ 1

Now one can apply Theorem A for the operator $H^1_{m(-\epsilon)}H^1_{m(-\epsilon)}$ in a analogous way as in Example 5.1 to investigate the optimal Orlicz domains.

As computation shows, in the case $0 \le \epsilon \le \frac{1}{m(-\epsilon)} - 1$ $0 \le \epsilon \le \frac{1}{m(-\epsilon)} - 1$ the optimality is attained as $\Omega\Omega$ ranges through $\mathcal{J}_{(1-\epsilon)}\mathcal{J}_{(1-\epsilon)}$. In the remaining examples there exists some $\Omega\Omega$

in $\mathcal{J}_{(1-\epsilon)}\mathcal{J}_{(1-\epsilon)}$ such that any Orlicz domain space in appropriate Sobolev embedding can be improved (see Table 2).

**5.3. Sobolev trace embeddings**

Our last application concerns the Sobolev trace embeddings.An open set $\Omega\Omega$ in $\mathbb{R}^{(1+2\epsilon)}\mathbb{R}^{(1+2\epsilon)}$ is said to have the cone property if there exists a finite cone $\Lambda\Lambda$ such that each point in $\Omega\Omega$ is the vertex of a finite cone contained in $\Omega\Omega$ and congruent to $\Lambda\Lambda$. Given an integer $(1+\epsilon)(1+\epsilon)$ such that $2\epsilon \geq \epsilon \geq 0, 2\epsilon \geq \epsilon \geq 0,$ we denote by $\Omega_{(1+\epsilon)}\Omega_{(1+\epsilon)}$ the nonempty intersection of $\Omega\Omega$ with a $(1+\epsilon)(1+\epsilon)$-dimensional affine subspace of $\mathbb{R}^{(1+2\epsilon)}.\mathbb{R}^{(1+2\epsilon)}$. The reduction principle for trace embeddings [9 Theorem 1.3] now has the following form. Let $\Omega\Omega$ be a bounded open set with cone property in $\mathbb{R}^{(1+2\epsilon)}, 2\epsilon \geq 1.\mathbb{R}^{(1+2\epsilon)}, 2\epsilon \geq 1$. Assume that $m \in \mathbb{N}$ $m \in \mathbb{N}$ and

$1+\epsilon \in \mathbb{N} 1+\epsilon \in \mathbb{N}$ are such that $2\epsilon \geq \epsilon \geq 0$ $2\epsilon \geq \epsilon \geq 0$ and $m \geq \epsilon. m \geq \epsilon$. Let $\|.\|_{X(0,1)}\|.\|_{X(0,1)}$ and $\|.\|_{Y(0,1)}\|.\|_{Y(0,1)}$be rearrangement-invariant function norms. Then the following facts are equivalent.

(i) The inequality $\left\| H_{\frac{1+\epsilon}{m}}^{\frac{1+2\epsilon}{1+2\epsilon}} \left(\sum f_j\right) \right\|_{Y(0,1)} \leq C \sum \|f_j\|_{X(0,1)}$ $\left\| H_{\frac{1+\epsilon}{m}}^{\frac{1+2\epsilon}{1+2\epsilon}} \left(\sum f_j\right) \right\|_{Y(0,1)} \leq C \sum \|f_j\|_{X(0,1)}$ holds for some constant $C$ $C$and for every nonnegative $f_j \in X(0,1) f_j \in X(0,1)$.

(ii) The Sobolev trace embedding $Tr: W^m X(\Omega) \to Y(\Omega_{1+\epsilon})$

$Tr: W^m X(\Omega) \to Y(\Omega_{1+\epsilon})$ holds.

**Table 3**

Application of Theorem A for the operator $H^{\frac{1+2\epsilon}{\frac{1+\epsilon}{m}}}_{1+2\epsilon} H^{\frac{1+2\epsilon}{\frac{1+\epsilon}{m}}}_{1+2\epsilon}$ and domain with cone property (see [20]).

| $Y(\Omega)$ | $L^B(\Omega)$ | $G(\Omega)$ |
|---|---|---|
| $L^{\frac{\epsilon^2+2\epsilon+1}{1+2\epsilon-m(1+\epsilon)}} \log^{\frac{\epsilon^2+2\epsilon+1}{1+2\epsilon-m(1+\epsilon)}} L(\Omega)$, $0 \le \epsilon < 2\epsilon$ | $L^{1+\epsilon}\log^{1+\epsilon} L(\Omega)$ | $(1+\epsilon)^{\frac{1+2\epsilon}{m-1+2\epsilon}+\frac{1+\epsilon}{\epsilon}} \log^{\frac{1+\epsilon}{\epsilon}} (1+\epsilon)$ |
| $L^{\frac{\epsilon^2+2\epsilon+1}{1+2\epsilon-m(1+\epsilon)}} \log^{\frac{\epsilon^2+2\epsilon+1}{1+2\epsilon-m(1+\epsilon)}} \log L(\Omega)$, $0 \le \epsilon < 2\epsilon$ | $L^{1+\epsilon}\log^{1+\epsilon}\log L(\Omega)$ | $(1+\epsilon)^{\frac{1+2\epsilon}{m-1+2\epsilon}+\frac{1+\epsilon}{\epsilon}} \log^{\frac{1+\epsilon}{\epsilon}} \log(1+\epsilon)$ |
| $\exp L^{\frac{1+2\epsilon}{1+2\epsilon-m(2+\epsilon)}}(\Omega)$, $\epsilon = 0$, $1 < \epsilon$ | $L^{1+\epsilon}\log^{1} L(\Omega)$ | $\log^{\frac{1+2\epsilon-m(2+\epsilon)}{1+2\epsilon-m}} (1+\epsilon)$ |
| $\exp\exp L^{\frac{2+\epsilon}{2+\epsilon-m}}(\Omega)$, $\epsilon = 0, \epsilon = 1$ | $L^{1+\epsilon}\log^{-(1+\epsilon)}\log L(\Omega)$ | $\log\log(1+\epsilon)$ |
| $\exp\left(L^{\frac{1+2\epsilon}{1+2\epsilon-m}} \log^{\frac{m(1+\epsilon)}{1+2\epsilon-m}} L\right)(\Omega)$ $\epsilon = 0$ | $L^{1+\epsilon}\log^{\epsilon} L \log^{1+\epsilon}\log L(\Omega)$ | $\log(1+\epsilon)\log^{\frac{m(1+\epsilon)}{m-1+2\epsilon}} \log (1+\epsilon)$ |
| $L^{\infty}(\Omega) L^{1+2\epsilon}(\Omega)$ | | $1$ |

Let $\Omega\Omega$ be a domain in $\mathbb{R}^{(1+2\epsilon)}\mathbb{R}^{(1+2\epsilon)}$ with cone property, $m \in \mathbb{N}, m < 1+2\epsilon, m \in \mathbb{N}, m < 1+2\epsilon$, and consider again the spaces $L^{1+\epsilon}\log^{1+\epsilon} L(\Omega)$ $L^{1+\epsilon}\log^{1+\epsilon} L(\Omega)$ o r $L^{1+\epsilon}\log^{1+\epsilon}\log L(\Omega), \epsilon > 0$ $L^{1+\epsilon}\log^{1+\epsilon}\log L(\Omega), \epsilon > 0$ and $1+\epsilon \in \mathbb{R} 1+\epsilon \in \mathbb{R}$ or $\epsilon = 0$ $\epsilon = 0$ and $\epsilon \ge -1 \epsilon \ge -1$. By[9, Theorem 5.2, Example 5.3

andExample 5.4] we have

$$Tr: W^m L^{1+\epsilon}\log^{1+\epsilon} L(\Omega) \to \begin{cases} L^{\frac{\epsilon^2+2\epsilon+1}{1+2\epsilon-m(1+\epsilon)}}\log^{\frac{\epsilon^2+2\epsilon+1}{1+2\epsilon-m(1+\epsilon)}} L_{(\Omega_{1+\epsilon})}, 0 \le \epsilon < 2\epsilon, \\ \exp L^{\frac{1+2\epsilon}{1+2\epsilon-m(2+\epsilon)}}(\Omega_{1+\epsilon}), \ \epsilon = 0, 0 < \epsilon, \\ \exp\exp L^{\frac{1+2\epsilon}{1+2\epsilon-m}}(\Omega_{1+\epsilon}), \ \epsilon = 0, \epsilon = 1, \\ L^\infty(\Omega_{1+\epsilon}), \qquad 0 > \epsilon \ \text{ or } \epsilon = 0, \ 1 > \epsilon, \end{cases}$$

and

$$Tr: W^m L^{1+\epsilon}\log^{1+\epsilon}\log L(\Omega) \to \begin{cases} L^{\frac{\epsilon^2+2\epsilon+1}{1+2\epsilon-m(1+\epsilon)}}\log^{\frac{\epsilon^2+2\epsilon+1}{1+2\epsilon-m(1+\epsilon)}} \log L_{(\Omega_{1+\epsilon})}, \quad 0 \le \epsilon < 2\epsilon, \\ \exp\left(L^{\frac{1+2\epsilon}{1+2\epsilon-m}}\log^{\frac{m(1+\epsilon)}{1+2\epsilon-m}} L\right)(\Omega_{1+\epsilon}), \quad \epsilon = 0, \\ L^\infty(\Omega_{1+\epsilon}), \qquad 0 > \epsilon, \end{cases}$$

and the range spaces being optimal in the class of Orlicz spaces.

Now, using Theorem A for the operator $H^{\frac{1+2\epsilon}{\frac{1+\epsilon}{m}}}H^{\frac{1+2\epsilon}{\frac{1+\epsilon}{m}}}$, one can investigate the optimal Orlicz domains. The situation is almost the same as in case of Sobolev embedding and hence we just present the results (see Table 3). Naturally, the optimality is attained only in the subcritied cases.

**5.4. Extension to other r.i. target spaces**

As we have seen in Example 5.1 a) in the case when the optimality is attained one can extend the positive result to other r.i. target spaces. Let us now look closer on this phenomenon. Let $(1-\epsilon)(1-\epsilon)$ and $(1+\epsilon)(1+\epsilon)$ be fixed and let $L^A(0,1)L^A(0,1)$ be an optimal Orlicz space rendering the relation $H^{1+\epsilon}_{1-\epsilon}: L^A(0,1) \to M(0,1)$ $H^{1+\epsilon}_{1-\epsilon}: L^A(0,1) \to M(0,1)$true, where $M(0,1)M(0,1)$is a given Marcinkiewicz endpoint space. We know from Theorem A that not every Orlicz space is an optimal domain space, such spaces are exactly those for which the supremum operator $(1-\epsilon)_{1-\epsilon}$ $(1-\epsilon)_{1-\epsilon}$ is bounded on their associate space. However, we can go the opposite direction. Suppose that $L^A(0,1)L^A(0,1)$ is a given Orlicz space such that the operator $(1-\epsilon)_{1-\epsilon}(1-\epsilon)_{1-\epsilon}$ is bounded on $L^{\tilde{A}}(0,1)L^{\tilde{A}}(0,1)$. Now thanks to the result of [17], the operator $(1-\epsilon)_{1-\epsilon}(1-\epsilon)_{1-\epsilon}$ is bounded on some r.i. space $X'(0,1)X'(0,1)$ if and only if the $X(0,1)X(0,1)$ is optimal r.i. domain space for some r.i. target space . By Proposition 3.5, the norm of the best r.i. target space, say $Y_{L^A}(0,1)Y_{L^A}(0,1)$, is given by

$$\|\Sigma f_j\|_{(Y_{L^A})^r(0,1)} = \left\|(1+\epsilon)^{-\epsilon}\int_0^{(1+\epsilon)^{\frac{1}{1+\epsilon}}}\Sigma f_j^*(1-\epsilon)d(1-\epsilon)\right\|_{L^{\tilde{A}}(0,1)}.$$

$$\|\Sigma f_j\|_{(Y_{L^A})^r(0,1)} = \left\|(1+\epsilon)^{-\epsilon}\int_0^{(1+\epsilon)^{\frac{1}{1+\epsilon}}}\Sigma f_j^*(1-\epsilon)d(1-\epsilon)\right\|_{L^{\tilde{A}}(0,1)}.$$

The fundamental function of $Y_{L^A}Y_{L^A}$ say $\varphi,\varphi$, then satisfies (cf. (3.9))

$$\varphi(1+\epsilon) \simeq (1+\epsilon)^{-\epsilon(1+\epsilon)}E^{1}_{-\epsilon}\left((1+\epsilon)^{-(1+\epsilon)}\right)$$

$$\simeq (1+\epsilon)^{-\epsilon(1+\epsilon)}\tilde{A}^{-1}\left((\epsilon^2+2\epsilon+1)^{-(1+\epsilon)}\right)$$

$$\simeq (1+\epsilon)^{-\epsilon(1+\epsilon)}\tilde{A}^{-1}\left((\epsilon^2+2\epsilon+1)^{-(1+\epsilon)}\right).$$

Moreover, to the given Orlicz space$L^A(0,1)L^A(0,1)$, we are able to compute the appropriate Marcinkiewicz space $M(0,1)M(0,1)$ . If we take a look at the proof of Theorem B again, we observe that in the case of optimality, (3.10) becomes actually equivalence, therefore the fundamental function of $M(0,1)M(0,1)$ is equivalent to $\varphi\varphi$. Consequently, we obtain that the space $L^A(0,1)L^A(0,1)$ is the optimal Orlicz for every r.i. space $Y(0,1)Y(0,1)$ satisfying $Y_{L^A}(0,1) \subseteq Y(0,1) \subseteq M(0,1)Y_{L^A}(0,1) \subseteq Y(0,1) \subseteq M(0,1)$.

**Example 5.2.** Let $\Omega\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^{(2+\epsilon)}, \epsilon \geq 0, \mathbb{R}^{(2+\epsilon)}, \epsilon \geq 0$, and $0 \leq \epsilon \leq 2 0 \leq \epsilon \leq 2$. One can easily observe that $(1-\epsilon)_{\frac{1}{1+\epsilon}}(1-\epsilon)_{\frac{1}{1+\epsilon}}$ is bounded on $L^{(1+\epsilon)'}(0,1)$ $L^{(1+\epsilon)'}(0,1)$, where $(1+\epsilon)' = \frac{1+\epsilon}{\epsilon}(1+\epsilon)' = \frac{1+\epsilon}{\epsilon}$. Then the optimal r.i. range space for the operator $H_{\frac{1}{1+\epsilon}}^{1} H_{\frac{1}{1+\epsilon}}^{1}$ is the Lorentz space $L^{(1+\epsilon)^*,(1+\epsilon)}(0,1)L^{(1+\epsilon)^*,(1+\epsilon)}(0,1)$, where

$0 < \epsilon < 2\epsilon, \epsilon \geq 0, (1+\epsilon)^* = \frac{2\epsilon^2+3\epsilon+1}{\epsilon}$

$0 < \epsilon < 2\epsilon, \epsilon \geq 0, (1+\epsilon)^* = \frac{2\epsilon^2+3\epsilon+1}{\epsilon}$. Its fundamental function is equivalent to the power function $(1+\epsilon)^{\frac{1}{(1+\epsilon)^*}}(1+\epsilon)^{\frac{1}{(1+\epsilon)^*}}$ and therefore, for every fixed $, 1 \leq \epsilon \leq \infty, 0 \leq \epsilon \leq \infty$ $, 1 \leq \epsilon \leq \infty, 0 \leq \epsilon \leq \infty$, the Lebesgue space $L^{1+\epsilon}(\Omega)L^{1+\epsilon}(\Omega)$ is the largest Orlicz space which renders the embedding $W^1L^{1+\epsilon}(\Omega) \hookrightarrow L^{(1+\epsilon)^*,2\epsilon}(\Omega)W^1L^{1+\epsilon}(\Omega) \hookrightarrow L^{(1+\epsilon)^*,2\epsilon}(\Omega)$ true. Similarly, for a given integer $0 < \epsilon < 2\epsilon\, 0 < \epsilon < 2\epsilon$ and ,, $0 \leq \epsilon \leq \infty 0 \leq \epsilon \leq \infty$, we obtain that $L^{1+\epsilon}(\Omega)L^{1+\epsilon}(\Omega)$ is the largest Orlicz space in

$$W^1L^{1+\epsilon}(\Omega) \hookrightarrow L^{\frac{\epsilon^2+3\epsilon+1}{-\epsilon^2},2\epsilon}(\Omega)W^1L^{1+\epsilon}(\Omega) \hookrightarrow L^{\frac{\epsilon^2+3\epsilon+1}{-\epsilon^2},2\epsilon}(\Omega).$$

## Conclusion

Finally we show that in general setting of rearrangement-invariant (r.i.) Banach function spaces, such questions were investigated using the method of reducing the Sobolev embeddings to the boundedness of an appropriate modification of the weighted Hardy operator.In the setting of r.i. spaces, the optimal domain and the optimal target spaces are then explicitly described. We have also present some defenitions and all the basic facts about Young functions and Orlicz Spaces. We use the easiest case of Sobolev embeddings, namely those acting on John domains. We present some applications in Sobolev embeddings on wider family of subsets so-called Maz'ya classes. Our last applications concerns the Sobolev trace embeddings.

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