On Some Basic Facts on Associated Functions and The Kernel Functions

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Abstract:

The study of associated functions and kernel functions occupies a central role in complex analysis and functional analysis, particularly in connection with Hilbert spaces of analytic function, reproducing kernels, and operator theory. This paper provides a concise survey of some basic facts on associated functions and kernel functions, with emphasis on their structural properties and interrelations. It followed the deductive method.We recall the definitions of associated functions in the framework of entire and analytic functions, highlight their connections to orthogonal systems, and analyze the role of kernel functions in reproducing kernel Hilbert spaces. Furthermore, we present illustrative examples that demonstrate how kernel functions encode analytic and geometric information about function space,while associated functions help in extending basses and characterizing. And the study found that the results are intended to clarify foundational aspects and to provide a reference point for further exploration of applications in approximation theory, spectral theory, and spaces of analytic functions.

Key words : Basic Facts , Kernel, analytic , Associated , Function

حول بعض الحقائق الأساسية عن الدوال المرتبطة و دوال النواة

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المستخلص:

تحتل دراسة الدوال المرتبطة و دوال النواة دورا محوريا في التحليل المركب و التحليل الدالي، ولاسيما فيما يتعلق بفضاءات هيلبرت للدوال التحليلية و نواة الإستنساخ ، و نظرية المؤثرات. تقدم هذه الورقة عرضا موجزا لبعض الحقائق الأساسية حول الدوال المرتبطة و دوال النواة ، مع التركيز على خصائصها البنيوية و العلاقات المتبادلة بينها. و اتبعت الدراسة المنهج الإستنباطي. نستعرض تعريفات الدوال المرتبطة في إطار الدوال التامة و التحليلية مع تسليط الضوء على إرتباطها بالأنظمة المتعامدة و تحليل دور دوال النواة في فضاءات هيلبرت ذات نواة الإستنساخ. و علاوة على ذلك ، تقدم أمثلة توضيحية تبين كيف تقوم دوال النواة بترميز المعلومات التحليلية و الهندسية المتعلقة بفضاء الدوال ، في حين تسهم الدوال المرتبطة في توسيع القواعد ووصف الخصائص البنيوية. تهدف النتائج إلى توضيح الجوانب الأساسية و توفير مرجع يمكن الإنطلاق منه لإستكشاف المزيد من التطبيقات في نظرية التقريب و نظرية الطيف و فضاءات الدوال التحليلية.

كلمات مفتاحية: حقائق اساسية، تحليلى ، نواة ، مترابطة ، دوال

Introduction:

We will introduce new methods and we determine the long-time dynamics of model including and finding conditions on the kernel functions that governs the onset analysis. Moreover for two important classes of kernel functions we show sharp estimates of kernel and associated functions. We obtain these sharp estimates by using subtle upper and lower solutions based on careful analysis of included kernel and associated functions.

Lemma 1 For $x \in \mathbb{R}^N/\{0\} x \in \mathbb{R}^N/\{0\}$ and $\psi > 0$,$\psi > 0$, let

$\Gamma_+(x,\psi) = \{y \in \partial B_\psi : y.x \geq 0\} \Gamma_+(x,\psi) = \{y \in \partial B_\psi : y.x \geq 0\}$
and $\Gamma_-(x,\psi) = \{y \in \partial B_\psi : y.x \leq 0\}$
$\Gamma_-(x,\psi) = \{y \in \partial B_\psi : y.x \leq 0\}$.

Therefore

$r = |r|, \tilde{H}_+(r,\psi) := \int_{\Gamma_+(x,\psi)} H(|x-y|)dS_y = \omega_{N-1}2^{3-N}\frac{\psi}{r^{N-2}}\int_{|\psi-r|}^{\sqrt{\psi^2+r^2}}([(\psi+r)^2-\varphi^2][\varphi^2-(\psi-r)^2])^{\frac{N-3}{2}}\varphi H(\varphi)d\varphi,$

$r = |r|, \tilde{H}_+(r,\psi) := \int_{\Gamma_+(x,\psi)} H(|x-y|)dS_y = \omega_{N-1}2^{3-N}\frac{\psi}{r^{N-2}}\int_{|\psi-r|}^{\sqrt{\psi^2+r^2}}([(\psi+r)^2-\varphi^2][\varphi^2-(\psi-r)^2])^{\frac{N-3}{2}}\varphi H(\varphi)d\varphi,$

$$\tilde{H}\_(r,\psi) := \int_{\Gamma\_(x,\psi)} H(|x-y|)dS_y$$

$$= \omega_{N-1}2^{3-N}\frac{\psi}{r^{N-2}}\int_{\sqrt{\psi^2+r^2}}^{\psi+r}([(\psi+r)^2-\varphi^2][\varphi^2-(\psi-r)^2])^{\frac{N-3}{2}}\varphi H(\varphi)\,d\varphi,$$

then

$\tilde{H}(r,\psi) = \omega_{N-1}2^{3-N}\frac{\psi}{r^{N-2}}\int_{|\psi-r|}^{\psi+r}([(\psi+r)^2-\varphi^2][\varphi^2-(\psi-r)^2])^{\frac{N-3}{2}}\varphi H(\varphi)\,d\varphi$

$\tilde{H}(r,\psi) = \omega_{N-1}2^{3-N}\frac{\psi}{r^{N-2}}\int_{|\psi-r|}^{\psi+r}([(\psi+r)^2-\varphi^2][\varphi^2-(\psi-r)^2])^{\frac{N-3}{2}}\varphi H(\varphi)\,d\varphi$

[4].

Proof. For any given $y \in \partial B_\psi y \in \partial B_\psi$ and $x \in \partial B_r x \in \partial B_r$ with $\psi, r > 0,\psi, r > 0$, let $\theta\theta$ denote the angle between yy and x,x, namely $y.x = \psi r\cos\theta, y.x = \psi r\cos\theta$, then let $S_\theta S_\theta$ denote the intersection of the hyperplane

$E_\theta ;= \{z \in \mathbb{R}^N : z.x = \psi\cos\theta,\} E_\theta ;= \{z \in \mathbb{R}^N : z.x = \psi\cos\theta,\}$ with the sphere $\partial B_\psi \partial B_\psi$ which clearly is an $N-2N-2$ dimensional sphere of radius $\rho\sin\theta.\rho\sin\theta$. Then

$H(|x-y|) \equiv J\left(\sqrt{\psi^2+r^2-2\psi r\cos\theta}\right)$

$H(|x-y|) \equiv J\left(\sqrt{\psi^2+r^2-2\psi r\cos\theta}\right)$ for $y \in S_\theta y \in S_\theta$,

and

$$\tilde{H}(r,\psi) = \int_{\partial B_\psi} (|x-y|)dS_y = \int_0^\pi H\left(\sqrt{\psi^2+r^2-2\psi r\cos\theta}\right)|S_\theta|\,\psi d\theta$$

$$= \int_0^\pi H\left(\sqrt{\psi^2+r^2-2\psi r\cos\theta}\right)\omega_{N-1}(\psi\sin\theta)^{N-2}\,\psi d\theta$$

$$= \omega_{N-1}\int_0^\pi \psi^{N-2}(\sin\theta)^{N-2} H\left(\sqrt{(\psi-r)^2-2r\psi(1-\cos\theta)}\right)d\theta$$

$$= \omega_{N-1}\int_0^\pi \psi^{N-2}\left[2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right]^{N-2} H\left(\sqrt{(\psi-r)^2-4r\psi\sin^2\left(\frac{\theta}{2}\right)}\right)d\theta$$

$$= 2^{N-1}\psi^{N-1}\omega_{N-1}\int_0^1 \xi^{N-2}(1-\xi^2)^{(N-3)/2} H\left(\sqrt{(\psi-r)^2-4r\psi\xi^2}\right)d\xi,$$

where we have used $\xi = \sin\left(\frac{\theta}{2}\right).\xi = \sin\left(\frac{\theta}{2}\right)$. The change of variable

$\varphi = \sqrt{(\psi-r)^2-4r\psi\xi^2}\varphi = \sqrt{(\psi-r)^2-4r\psi\xi^2}$ then gives

$$\tilde{H}(r,\psi) = \omega_{N-1}2^{3-N}\frac{\psi}{r^{N-2}}\int_{|\psi-r|}^{\psi+r}\left([(\psi+r)^2-\varphi^2][\varphi^2-(\psi-r)^2]\right)^{\frac{N-3}{2}}\varphi H(\varphi)\,d\varphi. \qquad (1)$$

Similarly, by the definition of $\widetilde{H}_+(r,\psi),\widetilde{H}_+(r,\psi)$, we obtain

$$\widetilde{H}_+(r,\psi) = \omega_{N-1}\int_0^{\pi/2}(\psi\sin\theta)^{N-2}H\left(\sqrt{\psi^2+r^2-2\psi r\cos\theta}\right)\psi d\theta$$

$$= 2^{N-1}\psi^{N-1}\omega_{N-1}\int_0^{\sqrt{2}/2}\xi^{N-2}(1-\xi^2)^{(N-3)/2}H\left(\sqrt{(\psi-r)^2-4r\psi\xi^2}\right)d\xi$$

$$= \omega_{N-1}2^{3-N}\frac{\psi}{r^{N-2}}\int_{|\psi-r|}^{\sqrt{\psi^2+r^2}}([(\psi+r)^2-\varphi^2][\varphi^2 - (\psi-r)^2])^{\frac{N-3}{2}}\varphi H(\varphi)\,d\varphi.$$

Analogously,

$$\widetilde{H}\_(r,\psi) = \omega_{N-1}\int_{\pi/2}^{\pi}(\psi\sin\theta)^{N-2}H\left(\sqrt{\psi^2+r^2-2\psi r\cos\theta}\right)\psi d\theta$$

$$= 2^{N-1}\psi^{N-1}\omega_{N-1}\int_{\sqrt{2}/2}^{1}\xi^{N-2}(1-\xi^2)^{(N-3)/2}H\left(\sqrt{(\psi-r)^2-4r\psi\xi^2}\right)d\xi$$

$$= \omega_{N-1}2^{3-N}\frac{\psi}{r^{N-2}}\int_{\sqrt{\psi^2+r^2}}^{\psi+r}([(\psi+r)^2-\varphi^2][\varphi^2-(\psi-r)^2])^{\frac{N-3}{2}}\varphi H(\varphi)\,d\varphi.$$

Define

$$\zeta_0(\varphi) = \begin{cases} 0, & \varphi < 0 \\ 2\varphi & \varphi \in [0,1] \\ 2 & \varphi > 1, \end{cases} \quad (2)(2) \qquad \zeta_0(\varphi) = \begin{cases} 0, & \varphi < 0 \\ 2\varphi & \varphi \in [0,1] \\ 2 & \varphi > 1, \end{cases}$$

and for $\varepsilon > 0\varepsilon > 0$ and $\iota \in \mathbb{R}, \iota \in \mathbb{R}$, define

$$H_\varepsilon(\iota) := \int_{\mathbb{R}^{N-1}} H(|(\varrho, y')|)[1 + \zeta_0(|(\varrho, y')| - \varepsilon^{-1})]\, dy'. \qquad (3)$$

Lemma 2 For any given small numbers $\delta > 0\delta > 0$ and $\varepsilon > 0$ $\varepsilon > 0$ [3],

$\tilde{H}_+(r,\psi) \le (1+\delta)H_\varepsilon(r-\psi)\tilde{H}_+(r,\psi) \le (1+\delta)H_\varepsilon(r-\psi)$

for $\psi \in \left[\frac{r}{2}, (1+\delta^2)r\right], r \ge (\delta\varepsilon)^{-1}$,

$\psi \in \left[\frac{r}{2}, (1+\delta^2)r\right], r \ge (\delta\varepsilon)^{-1}$,

where $\tilde{H}_+\tilde{H}_+$ is given by Lemma 1.

Proof. 1. Split of $H_\varepsilon(r-\psi).H_\varepsilon(r-\psi)$.

Denote

$G_\varepsilon(|x|) := H(|x|)[1 + \zeta_0(|x| - \varepsilon^{-1})]$
$G_\varepsilon(|x|) := H(|x|)[1 + \zeta_0(|x| - \varepsilon^{-1})]$ for $x \in \mathbb{R}^N.x \in \mathbb{R}^N$.

Then

$$H_\varepsilon(r-\psi) = \int_0^\infty \omega_{N-1}\xi^{N-2} G_\varepsilon\left(\sqrt{(r-\psi)^2 + \xi^2}\right) d\xi$$

$$= \omega_{N-1}\int_{|\psi-r|}^{\infty}[\varphi^2-(\psi-r)^2]^{\frac{N-3}{2}}\varphi G_{\varepsilon}(\varphi)\,d\varphi$$
$$+\ \omega_{N-1}\int_{|\psi-r|}^{\infty}[\varphi^2-(\psi-r)^2]^{\frac{N-3}{2}}\varphi H(\varphi)\,d\varphi$$
$$+\,\omega_{N-1}\int_{|\rho-r|}^{\infty}[\varphi^2-(\psi-r)^2]^{\frac{N-3}{2}}\varphi J(\varphi)\zeta_0(\varphi-\varepsilon^{-1})\,d\varphi$$
$$=: W_1+W_2.$$

2. Split of $\tilde{H}_+(r,\psi).\tilde{H}_+(r,\psi)$.

By Lemma 1, we have

$$\tilde{H}_+(r,\psi)=\omega_{N-1}2^{3-N}\frac{\psi}{r^{N-2}}\int_{|\psi-r|}^{\sqrt{(r-\psi)^2+4\delta^2r\psi}}([(\psi+r)^2-\varphi^2][\varphi^2$$
$$-(\psi-r)^2])^{\frac{N-3}{2}}\varphi H(\varphi)\,d\varphi$$
$$+\,\omega_{N-1}2^{3-N}\frac{\psi}{r^{N-2}}\int_{\sqrt{(r-\psi)^2+4\delta^2r\psi}}^{\sqrt{r^2+\psi^2}}([(\psi+r)^2-\varphi^2][\varphi^2$$
$$-(\psi-r)^2])^{\frac{N-3}{2}}\varphi H(\varphi)\,d\varphi =: Q_1+Q_2.$$

3. We prove $Q_1\le(1+\delta)W_1.Q_1\le(1+\delta)W_1$.
For $|\psi-r|\le\varphi\le\sqrt{(r-\psi)^2+4\delta^2r\psi}$,
$|\psi-r|\le\varphi\le\sqrt{(r-\psi)^2+4\delta^2r\psi}$, we have

$$[(\psi+r)^2-\varphi^2]^{\frac{N-3}{2}}\le\begin{cases}(4r\psi)^{\frac{N-3}{2}} & N\ge 3,\\ [4r\psi(1-\delta^2)]^{\frac{-1}{2}} & N=2.\end{cases}$$

Then by the definition of Q_1Q_1 and W_1W_1 we have , for

$\psi \in \left[\frac{r}{2},(1+\delta^2)r\right], \psi \in \left[\frac{r}{2},(1+\delta^2)r\right],$

$$Q_1 \le \omega_{N-1}2^{3-N}\frac{\psi}{r^{N-2}}\frac{(4r\psi)^{\frac{N-3}{2}}}{\sqrt{1-\delta^2}}\int_{|\psi-r|}^{\sqrt{(r-\psi)^2+4\delta^2 r\psi}}[\varphi^2 - (\psi-r)^2]^{\frac{N-3}{2}}\varphi H(\varphi)\,d\varphi$$

$$= \frac{\omega_{N-1}}{\sqrt{1-\delta^2}}\left(\frac{\psi}{r}\right)^{\frac{N-1}{2}}\int_{|\psi-r|}^{\sqrt{(r-\psi)^2+4\delta^2 r\psi}}[\varphi^2 - (\psi-\iota)^2]^{\frac{N-3}{2}}\varphi H(\varphi)\,d\varphi \le \frac{(1-\delta^2)^{\frac{N-1}{2}}}{\sqrt{1-\delta^2}}W_1 \le (1+\delta)W_1$$

$$= \frac{\omega_{N-1}}{\sqrt{1-\delta^2}}\left(\frac{\psi}{r}\right)^{\frac{N-1}{2}}\int_{|\psi-r|}^{\sqrt{(r-\psi)^2+4\delta^2 r\psi}}[\varphi^2 - (\psi-\iota)^2]^{\frac{N-3}{2}}\varphi H(\varphi)\,d\varphi \le \frac{(1-\delta^2)^{\frac{N-1}{2}}}{\sqrt{1-\delta^2}}W_1 \le (1+\delta)W_1$$

sine $\delta > 0\delta > 0$ is small [10].

4. We show $Q_2 \le W_2 Q_2 \le W_2$.

For $\sqrt{(r-\psi)^2 + 4\delta^2 r\psi} \le \varphi \le \sqrt{r^2+\psi^2}$ $\sqrt{(r-\psi)^2 + 4\delta^2 r\psi} \le \varphi \le \sqrt{r^2+\psi^2}$, we have

$$[(\psi+r)^2 - \varphi^2]^{\frac{N-3}{2}} \le \begin{cases} (4r\psi)^{\frac{N-3}{2}}, & N \ge 3, \\ (2r\psi)^{\frac{-1}{2}}, & N = 2. \end{cases}$$

Then by the the definitions of $Q_2 Q_2$ and $W_2 W_2$ we have, for

$\psi \in \left[\frac{r}{2},(1+\delta^2)r\right], \psi \in \left[\frac{r}{2},(1+\delta^2)r\right],$

$$Q_2 \le \omega_{N-1} 2^{3-N} \frac{\psi}{r^{N-2}} (4r\psi)^{\frac{N-3}{2}} \sqrt{2} \int_{\sqrt{(r-\psi)^2+4\delta^2 r\psi}}^{\sqrt{r^2+\psi^2}} [\varphi^2 - (\psi - r)^2]^{\frac{N-3}{2}} \varphi H(\varphi)\, d\varphi$$

$$= \omega_{N-1} \sqrt{2} \left(\frac{\psi}{r}\right)^{\frac{N-1}{2}} \sqrt{2} \int_{\sqrt{(r-\psi)^2+4\delta^2 r\psi}}^{\sqrt{r^2+\psi^2}} [\varphi^2 - (\psi - r)^2]^{\frac{N-3}{2}} \varphi H(\varphi)\, d\varphi$$

$$\le \omega_{N-1} 2 \int_{\max\{|\psi-r|,\sqrt{2}\delta r\}}^{\sqrt{r^2+\psi^2}} [\varphi^2 - (\psi - r)^2]^{\frac{N-3}{2}} \varphi H(\varphi)\, d\varphi.$$

By the other side,

$$W_2 \ge \omega_{N-1} 2 \int_{\max\{|\psi-r|,\varepsilon^{-1}+1\}}^{\infty} [\varphi^2 - (\psi - r)^2]^{\frac{N-3}{2}} \varphi H(\varphi)\, d\varphi.$$

Hence, $Q_2 \le W_2 Q_2 \le W_2$ provided that $\sqrt{2}\delta r \ge \varepsilon^{-1} + 1$, $\sqrt{2}\delta r \ge \varepsilon^{-1} + 1$, namely

$$r \ge R := \frac{\varepsilon^{-1}+1}{\sqrt{2}\delta} \in (0, (\delta\varepsilon)^{-1}) .r \ge R := \frac{\varepsilon^{-1}+1}{\sqrt{2}\delta} \in (0, (\delta\varepsilon)^{-1}) .$$

The proof is complete [9].

Proposition 2. Let HH has compact support, say $\mathrm{supp}(H) \subset [0, K_*]\mathrm{supp}(H) \subset [0, K_*]$ for some $K_* > 0, K_* > 0$, and

$H_* H_*$ is given by 1.9. Then there exist constants $L_0 > 0 L_0 > 0$ and $C > 0 C > 0$ such that for $r \ge L_0, r \ge L_0$,

$$\left|\tilde{H}(r,\psi) - H_*(r-\psi)\right| \le C r^{-1} \qquad \text{when } \psi \in [r - K_*, r + K_*],$$

$$\left|\tilde{H}(r,\psi) = H_*(r-\psi)\right| = 0 \qquad \text{when } \psi \notin [r-K_*, r+K_*].$$

Proof. By using Lemma 1. and Lemma 2. we get

$$\tilde{H}(r,\psi) = \omega_{N-1} 2^{3-N} \frac{\psi}{r^{N-2}} \int_{|\psi-r|}^{\psi+r} ([(\psi+r)^2 - \varphi^2][\varphi^2 - (\psi-r)^2])^{\frac{N-3}{2}} \varphi H(\varphi)\, d\varphi,$$

$$H_*(r-\psi) = \omega_{N-1} \int_{|\psi-r|}^{\infty} [\varphi^2 - (\psi-r)^2]^{\frac{N-3}{2}} \varphi H(\varphi)\, d\varphi$$

Then,

$\tilde{H}(r,\psi) = H_*(r-\psi) = 0 \tilde{H}(r,\psi) = H_*(r-\psi) = 0$ when $|r-\psi| > K_*, |r-\psi| > K_*$, namely when $\psi \notin [r-K_*, r+K_*]$. $\psi \notin [r-K_*, r+K_*]$.

Moreover, for $r > K_* r > K_*$ and

$|r-\psi| > K_*, \tilde{H}(r,\psi) = H_*(r-\psi) \le \omega_{N-1} \int_{|\psi-r|}^{K_*} \left| \frac{2^{3-N}\psi}{r^{N-2}} ((\psi+r)^2 - \varphi^2)^{\frac{N-3}{2}} - 1 [\varphi^2 - (\psi-r)^2]^{\frac{N-3}{2}} \right| \varphi H(\varphi)\, d\varphi \le \omega_{N-1} \|H\|_\infty M(r) \int_{|\psi-r|}^{K_*} [\varphi^2 - (\psi-r)^2]^{\frac{N-3}{2}} \varphi H(\varphi)$

$|r-\psi| > K_*, \tilde{H}(r,\psi) = H_*(r-\psi) \le \omega_{N-1} \int_{|\psi-r|}^{K_*} \left| \frac{2^{3-N}\psi}{r^{N-2}} ((\psi+r)^2 - \varphi^2)^{\frac{N-3}{2}} - 1 [\varphi^2 - (\psi-r)^2]^{\frac{N-3}{2}} \right| \varphi H(\varphi)\, d\varphi \le \omega_{N-1} \|H\|_\infty M(r) \int_{|\psi-r|}^{K_*} [\varphi^2 - (\psi-r)^2]^{\frac{N-3}{2}} \varphi H(\varphi)$

$$M(r) := \max_{\{\psi \in [r-K_*, r+K_*], \varphi \in [0,K_*]\}} \left| \frac{2^{3-N}\psi}{r^{N-2}} ((\psi+r)^2 - \varphi^2)^{\frac{N-3}{2}} - 1 \right|$$

$$= \max_{\{\psi\in[r-K_*,r+K_*],\varphi\in[0,K_*]\}}\left|2^{3-N}\frac{\psi}{r}\left(\left(1+\frac{\psi}{r}\right)^2-\left(\frac{\varphi}{r}\right)^2\right)^{\frac{N-3}{2}}-1\right|$$

$$= \max_{\{\xi\in[-K_*,K_*],\varphi\in[0,K_*]\}}\left|\left(1+\frac{\xi}{r}\right)\left[\left(1+\frac{\xi}{2r}\right)^2-\left(\frac{\varphi}{2r}\right)^2\right]^{\frac{N-3}{2}}-1\right| = O(r^{-1})$$

$$= \max_{\{\xi\in[-K_*,K_*],\varphi\in[0,K_*]\}}\left|\left(1+\frac{\xi}{r}\right)\left[\left(1+\frac{\xi}{2r}\right)^2-\left(\frac{\varphi}{2r}\right)^2\right]^{\frac{N-3}{2}}-1\right| = O(r^{-1})$$

as $r \to \infty$,$r \to \infty$,

and

$$\int_{|\psi-r|}^{K_*}[\varphi^2-(\psi-r)^2]^{\frac{N-3}{2}}\varphi\, d\varphi$$

$$\leq \begin{cases} \int_0^{K_*}\varphi^{N-2}d\varphi & \text{if } N\geq 3, \\ \int_{|\psi-r|}^{K_*}\frac{\varphi}{\sqrt{\varphi+|\psi-r|}}\frac{1}{\sqrt{\varphi-|\psi-r|}}d\varphi \leq \int_0^{K_*}\frac{\sqrt{K_*}}{\sqrt{\xi}}d\xi & \text{if } N=2. \end{cases}$$

Therefore there exists $C>0C>0$ such that

$\tilde{H}(r,\psi) = H_*(r-\psi) \leq Cr^{-1}\tilde{H}(r,\psi) = H_*(r-\psi) \leq Cr^{-1}$ for $|\psi - r| \leq K_*|\psi - r| \leq K_*$ and all large rr.

The proof is complete [5].

Lemma 4. If **(H1)(H1)** holds, then for any $\varepsilon \in (0,1)$,$\varepsilon \in (0,1)$,

$$\lim_{R\to\infty}\int_0^{(1-\varepsilon)R}\int_R^{\infty}\widetilde{H}_+(r,\psi)d\psi dr = 0 \lim_{R\to\infty}\int_0^{R}\int_R^{\infty}\widetilde{H}_-(r,\psi)d\psi dr = 0. \quad (4)$$

Proof . For $R \geq r > 0, R \geq r > 0$, denote

$x_{r:=}^1(r,0,\dots,0) \in \mathbb{R}^N x_{r:=}^1(r,0,\dots,0) \in \mathbb{R}^N$ and
$\Omega_R^+ := \{y = (y_1,\dots,y_N) : y_1 > 0, |y| > R\}$.
$\Omega_R^+ := \{y = (y_1,\dots,y_N) : y_1 > 0, |y| > R\}$.

Therefore

$\int_R^{\infty}\widetilde{H}_+(r,\psi)d\psi = \int_{\Omega_R^+} H(|x_r^1 - y|)dy$.
$\int_R^{\infty}\widetilde{H}_+(r,\psi)d\psi = \int_{\Omega_R^+} H(|x_r^1 - y|)dy$. For small $\delta \in (0,\varepsilon)$
$\delta \in (0,\varepsilon)$ define

$Q_R^\delta := \{z = (z_1, z_2, \dots, z_N) : z_1 \leq (1-\delta)R, |z_i| \leq \Lambda R, 2 \leq i \leq N\}$

with $\Lambda := \frac{\sqrt{1-(1-\delta)^2}}{\sqrt{N}} \Lambda := \frac{\sqrt{1-(1-\delta)^2}}{\sqrt{N}}$. Clearly,
$\frac{\Omega_R^+ \subset \mathbb{R}^N}{Q_R^\delta}$.

Then

$$\int_R^{\infty}\widetilde{H}_+(r,\psi)d\psi = \int_{\Omega_R^+} H(|x_r^1 - y|)dy \leq \int_{\frac{\mathbb{R}^N}{Q_R^\delta}} H(|x_r^1 - y|)dy.$$

The set $\frac{\mathbb{R}^N \mathbb{R}^N}{Q_R^\delta Q_R^\delta}$ can be analyzed as follows:

$$\frac{\mathbb{R}^N}{Q_R^\delta} = \bigcup_{i=1}^{N} S^{(i)}$$

with overlapping sets

$$S^{(1)} := \{z = (z_1, z_2, \dots, z_N) : z_1 > (1-\delta)R \text{ and } z_i \in \mathbb{R} \text{ for } 2 \le i \le N\}$$

$$S^{(j)} := \{z = (z_1, z_2, \dots, z_N) : |z_j| > \Lambda R \text{ and } z_i \in \mathbb{R} \text{ for } i \ne j\}\ 2 \le i \le N.$$

Therefore, making use of the definition of H_*,H_*, we deduce

$$\int_R^\infty \tilde{H}_+(r,\psi)d\psi \le \int_{\frac{\mathbb{R}^N}{Q_R^\delta}} H(|x_r^1 - y|)dy \le \sum_{j=1}^{N} \int_{S^{(j)}} H(|x_r^1 - y|)dy$$

$$= \int_{(1-\delta)R}^\infty H_*(r-\psi)d\psi + 2(N-1)\int_{\Lambda R}^\infty H_*(\psi)d\psi$$

$$= \int_{(1-\delta)R-r}^\infty H_*(\xi)d\xi + 2(N-1)\int_{\Lambda R}^\infty H_*(\psi)d\psi.$$

It follows that

$$\int_0^{(1-\varepsilon)R}\int_R^\infty \tilde{H}_+(r,\psi)d\psi dr$$

$$\le \int_0^{(1-\varepsilon)R}\int_{(1-\delta)R-r}^\infty H_*(\xi)d\xi + 2(N-1)(1-\varepsilon)R\int_{\Lambda R}^\infty H_*(\psi)d\psi.$$

We have, due to **(H1)(H1)**,

$$\int_0^{(1-\varepsilon)R}\int_{(1-\delta)R-r}^{\infty} H_*(\xi)d\xi dr$$
$$\leq \int_0^{(1-\varepsilon)R}\int_{(1-\delta)R}^{\infty}[\xi-(\varepsilon-\delta)R]H_*(\xi)d\xi$$
$$\leq \int_{(1-\delta)R}^{\infty}\xi H_*(\xi)d\xi \to 0 \text{ as } R\to\infty,$$

and

$2(N-1)(1-\varepsilon)R\int_{\Lambda R}^{\infty}H_*(\psi)d\psi \leq 2(N-1)(1-\varepsilon)\frac{1}{\Lambda}\int_{\Lambda R}^{\infty}\psi H_*(\psi)d\psi \to 0$ as $R\to\infty$

$2(N-1)(1-\varepsilon)R\int_{\Lambda R}^{\infty}H_*(\psi)d\psi \leq 2(N-1)(1-\varepsilon)\frac{1}{\Lambda}\int_{\Lambda R}^{\infty}\psi H_*(\psi)d\psi \to 0$ as $R\to\infty$

[1].

Then

$$\int_0^{(1-\varepsilon)R}\int_R^{\infty}\widetilde{H}_+(r,\psi)d\psi dr \to 0 \text{ as } R\to\infty.$$

Similarly, for $R \geq r > 0, R \geq r > 0,$

$x_r^1 := (r,0,\dots,0) \in \mathbb{R}^N x_r^1 := (r,0,\dots,0) \in \mathbb{R}^N$ and

$\Omega_R^- := \{y=(y_1,\dots,y_N) : y_1 < 0, |y| > R\},$

$\Omega_R^- := \{y=(y_1,\dots,y_N) : y_1 < 0, |y| > R\},$

we have

$$\int_R^{\infty}\widetilde{H}\_(r,\psi)d\psi = \int_{\Omega_R^-}H(|x_r^1-y|)\,dy.$$

Suppose

$$\tilde{Q}_R := \left\{ z = (z_1, z_2, \dots, z_N) : z_1 \geq \frac{R}{2\sqrt{N}}, |z_i| \leq \frac{R}{2\sqrt{N}}, 2 \leq i \leq N \right\}.$$

Hence

$$\Omega_R^- \subset \frac{\mathbb{R}^N}{\tilde{Q}_R} = \bigcup_{j=1}^{N} \tilde{S}^{(j)}$$

with

$$\tilde{S}^{(1)} := \left\{ z = (z_1, z_2, \dots, z_N) : z_1 \leq -\frac{R}{2\sqrt{N}}, z_i \in \mathbb{R} \ \text{ for } i \neq 1 \right\}$$

$$\tilde{S}^{(j)} := \left\{ z = (z_1, z_2, \dots, z_N) : |z_j| \geq \frac{R}{2\sqrt{N}} \text{ and } z_i \in \mathbb{R} \ \text{ for } i \neq j \right\} \ 2 \leq i \leq N.$$

Moreover

$$\int_R^{\infty} \tilde{H}\_(r,\psi)d\psi = \int_{\Omega_R^-} H(|x_r^1 - y|)\, dy \leq \sum_{j=1}^{N} \int_{(j)} H(|x_r^1 - y|)\, dy$$

$$\leq \int_{-\infty}^{\frac{R}{2\sqrt{N}}} H_*(r-\psi)d\psi$$

$$+ 2(N-1)\int_{\frac{R}{2\sqrt{N}}}^{\infty} H_*(\psi)d\psi = \int_{\frac{R}{2\sqrt{N}}+r}^{\infty} H_*(\xi)d\xi$$

$$+ 2(N-1)\int_{\frac{R}{2\sqrt{N}}}^{\infty} H_*(\psi)d\psi.$$

It is follows that

$$\int_0^R \int_R^\infty \tilde{H}_-(r,\psi)d\psi dr \le \int_0^R \int_{\frac{R}{2\sqrt{N}}+r}^\infty H_*(\xi)d\xi dr + 2(N-1)\,R \int_{\frac{R}{2\sqrt{N}}}^\infty H_*(\psi)d\psi \le$$
$$\int_{\frac{R}{2\sqrt{N}}+r}^\infty \xi H_*(\xi)d\xi + 4(N-1)\sqrt{N}\int_{\frac{R}{2\sqrt{N}}}^\infty \psi H_*(\psi)d\psi \to 0 \text{ as } R \to \infty$$

$$\int_0^R \int_R^\infty \tilde{H}_-(r,\psi)d\psi dr \le \int_0^R \int_{\frac{R}{2\sqrt{N}}+r}^\infty H_*(\xi)d\xi dr + 2(N-1)\,R \int_{\frac{R}{2\sqrt{N}}}^\infty H_*(\psi)d\psi \le$$
$$\int_{\frac{R}{2\sqrt{N}}+r}^\infty \xi H_*(\xi)d\xi + 4(N-1)\sqrt{N}\int_{\frac{R}{2\sqrt{N}}}^\infty \psi H_*(\psi)d\psi \to 0 \text{ as } R \to \infty$$

[6].

Lemma 5. If $(\mathbf{H1})(\mathbf{H1})$ holds, then

$$\lim_{R\to\infty} \int_0^R \int_R^\infty \tilde{H}(r,\psi)d\psi dr = \int_0^\infty \varphi H_*(\varphi)d\varphi\,.$$

Proof. We complete the proof in two steps.

$$\limsup_{R\to\infty} \int_0^R \int_R^\infty \tilde{H}(r,\psi)d\psi dr \le \int_0^\infty \varphi H_*(\varphi)d\varphi\,. \qquad (5)$$

By Lemma 4, for any small $\varepsilon_1 > 0\varepsilon_1 > 0$,

$$\limsup_{R\to\infty} \int_0^R \int_R^\infty \tilde{H}(r,\psi)d\psi dr = \limsup_{R\to\infty} \int_{(1-\varepsilon_1)R}^R \int_R^\infty \tilde{H}_+(r,\psi)d\psi dr\,,$$

and

$$\limsup_{R\to\infty} \int_{(1-\varepsilon_1)R}^R \int_{(1+\varepsilon_1)R}^\infty \tilde{H}_+(r,\psi)d\psi dr \le \lim_{\tilde{R}\to\infty} \int_0^{(1+\varepsilon_1)^{-1}R} \int_{\tilde{R}}^\infty \tilde{H}_+(r,\psi)d\psi dr$$
$$= 0\,.$$

Therefore

$$\limsup_{R\to\infty}\int_0^R\int_R^\infty \widetilde{H}(r,\psi)d\psi dr$$
$$=\limsup_{R\to\infty}\int_{(1-\varepsilon_1)R}^R\int_R^{(1+\varepsilon_1)R}\widetilde{H}_+(r,\psi)d\psi dr\,. \quad (6)$$

By Lemma 2, for any small $\delta > 0\delta > 0$ and $\varepsilon > 0,\varepsilon > 0$, we have

$\widetilde{H}_+(r,\psi) \le (1+\delta)J_\varepsilon(r-\psi)$ for $\psi \in \left[\frac{r}{2},(1+\delta^2)r\right]$
$\widetilde{H}_+(r,\psi) \le (1+\delta)J_\varepsilon(r-\psi)$ for $\psi \in \left[\frac{r}{2},(1+\delta^2)r\right]$ amd
$r \ge (\delta\varepsilon)^{-1}\,. \quad (7) r \ge (\delta\varepsilon)^{-1}\,. \quad (7)$

Therefore if $\varepsilon_1 > 0\varepsilon_1 > 0$ is sufficiently small, then $(7)(7)$ holds when

$(1-\varepsilon_1)R \le r \le R \le \psi \le (1+\varepsilon_1)R$
$(1-\varepsilon_1)R \le r \le R \le \psi \le (1+\varepsilon_1)R$ and $R = 2(\delta\varepsilon)^{-1}$
$R = 2(\delta\varepsilon)^{-1}$ [2]

we obtain

$$\int_{(1-\varepsilon_1)R}^R\int_R^{(1+\varepsilon_1)R}\widetilde{H}_+(r,\psi)d\psi dr \le (1+\delta)\int_{(1-\varepsilon_1)R}^R\int_R^{(1+\varepsilon_1)R}H_\varepsilon(r-\psi)d\psi dr\,.$$

From $(3)(3)$ we have

$H_\varepsilon(\varrho) = H_*(\varrho) + \xi_\varepsilon^*(\varrho)H_\varepsilon(\varrho) = H_*(\varrho) + \xi_\varepsilon^*(\varrho)$ with

$$\xi_\varepsilon^*(\varrho) := \int_{\mathbb{R}^{N-1}} H[(|(\varrho,y')|)\zeta_0(|(\varrho,y')| - \varepsilon^{-1})]\,dy'.$$
$$\xi_\varepsilon^*(\varrho) := \int_{\mathbb{R}^{N-1}} H[(|(\varrho,y')|)\zeta_0(|(\varrho,y')| - \varepsilon^{-1})]\,dy'.$$

Clearly

$$\int_{(1-\varepsilon_1)R}^{R}\int_{R}^{(1+\varepsilon_1)R} H_*(r-\psi)d\psi dr = \int_{-\varepsilon_1 R}^{0}\int_{0}^{\varepsilon_1 R} H_*(r-\psi)d\psi dr$$
$$\leq \int_{-\infty}^{0}\int_{0}^{\infty} H_*(r-\psi)d\psi dr = \int_{0}^{\infty} \varphi H_*(\varphi)d\varphi .$$

It is follows that

$$\int_{(1-\varepsilon_1)R}^{R}\int_{R}^{(1+\varepsilon_1)R} \widetilde{H}_+(r,\psi)d\psi dr \leq (1$$
$$+\delta)\left[\int_{0}^{\infty} \varphi H_*(\varphi)d\varphi + \int_{(1-\varepsilon_1)R}^{R}\int_{R}^{(1+\varepsilon_1)R} \xi_\varepsilon^*(r-\psi)d\psi dr\right].$$

Moreover, using $\varepsilon^{-1} = \frac{1}{2}\delta R \varepsilon^{-1} = \frac{1}{2}\delta R$ and
$\xi_\varepsilon^*(s) \leq 2H_*(s), \xi_\varepsilon^*(s) = 0$ for $s \leq \varepsilon^{-1}$,
$\xi_\varepsilon^*(s) \leq 2H_*(s), \xi_\varepsilon^*(s) = 0$ for $s \leq \varepsilon^{-1}$, we obtain

$$\int_{(1-\varepsilon_1)R}^{R}\int_{R}^{(1+\varepsilon_1)R} \xi_\varepsilon^*(r-\psi)d\psi dr$$
$$= \int_{-\varepsilon_1 R}^{0}\int_{0}^{\varepsilon_1 R} \xi_\varepsilon^*(r-\psi)d\psi dr$$
$$\leq 2\int_{-\varepsilon_1 R}^{0}\int_{\varepsilon^{-1}}^{\varepsilon_1 R - r} H_*(s)ds dr \leq 2\,\varepsilon_1 R \int_{\frac{1}{2}\delta R}^{\infty} H_*(s)ds$$
$$\leq 4\varepsilon_1\delta^{-1}\int_{\frac{1}{2}\delta R}^{\infty} sH_*(s)ds \to 0 \text{ as } R \to \infty.$$

We thus get

$$\limsup_{R\to\infty} \int_{(1-\varepsilon_1)R}^{R} \int_{R}^{(1+\varepsilon_1)R} H_{+}(r,\psi)d\psi dr \leq (1+\delta)\int_0^{\infty} \varphi H_{*}(\varphi)d\varphi.$$

Since $\delta > 0\delta > 0$ can be arbitrarily small, $(5)(5)$ now follows $(6).(6)$.

Step 2. We show that

$$\liminf_{R\to\infty} \int_0^{R} \int_{R}^{\infty} \tilde{H}(r,\psi)d\psi dr \geq \int_0^{\infty} \varphi H_{*}(\varphi)d\varphi.$$

The proof of the Lemma will be finishes [7].

From the definition of $\tilde{H}(r,\psi)\tilde{H}(r,\psi)$ and H_{*}, H_{*}, for $R \geq r > 0\ ,R \geq r > 0$,

$$\int_R^{\infty} \tilde{H}(r,\psi)d\psi \geq \int_{H_R} H(|x_r^1 - y|)dy = \int_0^{R} H_{*}(r-\psi)d\psi, \qquad (8)$$

where

$$x_r^1 := (r,0,\dots,0) \in \mathbb{R}^N, H_R : \\ = \{y = (y_1, y_2, \dots, y_N) : y_1 > R, y_i \in \mathbb{R} \text{ for } 2 \leq i \leq N\}.$$

Therefore

$$\lim_{R\to\infty}\inf\int_0^R\int_R^\infty \widetilde{H}(r,\psi)d\psi dr$$
$$\geq \lim_{R\to\infty}\int_0^R\int_R^\infty (r-\psi H_*)d\psi dr$$
$$= \lim_{R\to\infty}\int_0^R \varphi H_*(\varphi)d\varphi = \int_0^\infty \varphi H_*(\varphi)d\varphi.$$

The proof is now complete[11] .

Theorem 6. Assume **(H)(H)** holds. Then the following statements are equivalent :

(i) **(H1)(H1)** holds, namely $\int_0^\infty r^N H(r)\,dr < \infty$, $\int_0^\infty r^N H(r)\,dr < \infty$,

(ii) $\int_0^\infty H_*(\varrho)\varrho\,d\varrho < \infty.\int_0^\infty H_*(\varrho)\varrho\,d\varrho < \infty.$

(iii) $\lim_{R\to\infty}\sup\int_0^R\int_R^{+\infty}\widetilde{H}(r,\psi)d\psi dr < \infty$ $\lim_{R\to\infty}\sup\int_0^R\int_R^{+\infty}\widetilde{H}(r,\psi)d\psi dr < \infty$

(iv) $\lim_{R\to\infty}\sup\int_0^R\int_R^{+\infty}\left(\frac{r}{R}\right)^{N-1}\widetilde{H}(r,\psi)d\psi dr < \infty.$ $\lim_{R\to\infty}\sup\int_0^R\int_R^{+\infty}\left(\frac{r}{R}\right)^{N-1}\widetilde{H}(r,\psi)d\psi dr < \infty.$

Moreover, when **(H1)(H1)** holds, we have

$$\int_0^\infty H_*(\varrho)\varrho d\varrho$$
$$= \lim_{R\to\infty}\int_0^R\int_R^{+\infty}\widetilde{H}(r,\psi)d\psi dr$$
$$= \lim_{R\to\infty}\int_0^R\int_R^{+\infty}\left(\frac{r}{R}\right)^{N-1}\widetilde{H}(r,\psi)d\psi dr.$$

Proof. By

$\int_0^\infty H_*(\varrho)\varrho d\varrho = \frac{\omega_{N-1}}{N-1}\int_0^\infty H(r)\, r^N dr$
$\int_0^\infty H_*(\varrho)\varrho d\varrho = \frac{\omega_{N-1}}{N-1}\int_0^\infty H(r)\, r^N dr$ and Lemma 5 we see that

$(\mathbf{H1})(\mathbf{H1})$ holds, $\Leftrightarrow \int_0^\infty H_*(\varrho)\varrho d\varrho < \infty$,
$\Leftrightarrow \int_0^\infty H_*(\varrho)\varrho d\varrho < \infty$,

$(\mathbf{H1})(\mathbf{H1})$ holds, $\Rightarrow \lim_{R\to\infty}\int_0^R \int_R^{+\infty} \tilde{H}(r,\psi)d\psi dr < \infty$,
$\Rightarrow \lim_{R\to\infty}\int_0^R \int_R^{+\infty} \tilde{H}(r,\psi)d\psi dr < \infty$,

and if $(\mathbf{H1})(\mathbf{H1})$ holds, then

$$\int_0^\infty H_*(\varrho)\varrho d\varrho = \lim_{R\to\infty}\int_0^R \int_R^{+\infty} \tilde{H}(r,\psi)d\psi dr. \qquad (9)$$

To finish the proof of Theorem 6, it remains to prove that

$(\mathbf{H1})(\mathbf{H1})$ holds,

$\Leftarrow \lim_{R\to\infty}\sup\int_0^R \int_R^{+\infty} \tilde{H}(r,\psi)d\psi dr < \infty$, (10)
$\Leftarrow \lim_{R\to\infty}\sup\int_0^R \int_R^{+\infty} \tilde{H}(r,\psi)d\psi dr < \infty$, (10)

$(\mathbf{H1})(\mathbf{H1})$ holds,

$\Leftrightarrow \lim_{R\to\infty}\sup\int_0^R \left(\frac{r}{R}\right)^{N-1} \int_R^{+\infty} \tilde{H}(r,\psi)d\psi dr.$ (11)
$\Leftrightarrow \lim_{R\to\infty}\sup\int_0^R \left(\frac{r}{R}\right)^{N-1} \int_R^{+\infty} \tilde{H}(r,\psi)d\psi dr.$ (11)

and

$$\lim_{R\to\infty}\int_0^R\left(\frac{r}{R}\right)^{N-1}\int_R^{+\infty}\tilde{H}(r,\psi)d\psi dr=\int_0^\infty H_*(\varrho)\varrho d\varrho \qquad (12)$$

if $(\mathbf{H1})(\mathbf{H1})$ holds.

We now prove these in three steps.

Step 1. We prove $(10)(10)$.

By $(8)(8)$ and change of order of integration

$$\int_0^R\int_R^\infty \tilde{H}(r,\psi)d\psi dr\geq\int_{-R}^0\int_0^\infty H_*\,(r-\psi)d\psi dr\geq\int_0^\infty \varrho H_*(\varrho)\varrho d\varrho\,,$$

which yields [1]

$$\infty>\limsup_{R\to\infty}\int_0^R\int_R^\infty \tilde{H}(r,\psi)d\psi dr\geq\int_0^\infty \varrho H_*(\varrho)\varrho d\varrho\,.$$

Hence, due to $(11)(11)$, $(10)(10)$ holds.

Step 2. We prove $(11).(11)$.

If $(\mathbf{H1})(\mathbf{H1})$ holds then

$$\limsup_{R\to\infty}\int_0^R\left(\frac{r}{R}\right)^{N-1}\int_R^{+\infty}\tilde{H}(r,\psi)d\psi dr$$
$$\leq\limsup_{R\to\infty}\int_0^R\int_R^{+\infty}\tilde{H}(r,\psi)d\psi dr<\infty. \qquad (13)$$

On the other hand, if

$$\limsup_{R\to\infty}\int_0^R\left(\frac{r}{R}\right)^{N-1}\int_R^{+\infty}\tilde{H}(r,\psi)d\psi dr<\infty,$$

then by (8)(8),

$$\infty > \lim_{R\to\infty} \sup \int_{\frac{R}{2}}^{R} \left(\frac{r}{R}\right)^{N-1} \int_{R}^{+\infty} \tilde{H}(r,\psi)d\psi dr$$

$$\geq \lim_{R\to\infty} \sup 2^{-(N-1)} \int_{\frac{R}{2}}^{R} \int_{R}^{+\infty} H_*(r-\psi)d\psi dr$$

$$= \lim_{R\to\infty} \sup 2^{-(N-1)} \int_{0}^{-\frac{R}{2}} \int_{R}^{+\infty} H_*(r-\psi)d\psi dr$$

$$\geq \lim_{R\to\infty} \sup 2^{-(N-1)} \int_{0}^{-\frac{R}{2}} \varrho H_*(\varrho)d\varrho.$$

Hence(11)(11) holds [8].

Step 3. We finally prove (12)(12).

For any given $\varepsilon > 0$,$\varepsilon > 0$, we have

$$\int_{0}^{R} \left(\frac{r}{R}\right)^{N-1} \int_{R}^{+\infty} \tilde{H}(r,\psi)d\psi dr \geq \int_{(1-\varepsilon)R}^{R} \left(\frac{r}{R}\right)^{N-1} \int_{R}^{+\infty} \tilde{H}(r,\psi)d\psi dr$$

$$\geq (1-\varepsilon)^{N-1} \int_{(1-\varepsilon)R}^{R} \int_{R}^{+\infty} \tilde{H}(r,\psi)d\psi dr.$$

By (8)(8)

$$\int_{(1-\varepsilon)R}^{R} \int_{R}^{+\infty} \tilde{H}(r,\psi)d\psi dr \geq \int_{(1-\varepsilon)R}^{R} \int_{R}^{\infty} H_*(r-\psi)d\psi dr = \int_{-\varepsilon R}^{0} \int_{0}^{\infty} H_*(r-\psi)d\psi dr \geq \int_{0}^{\varepsilon R} \varrho H_*(\varrho)d\varrho$$

$$\int_{(1-\varepsilon)R}^{R} \int_{R}^{+\infty} \tilde{H}(r,\psi)d\psi dr \geq \int_{(1-\varepsilon)R}^{R} \int_{R}^{\infty} H_*(r-\psi)d\psi dr = \int_{-\varepsilon R}^{0} \int_{0}^{\infty} H_*(r-\psi)d\psi dr \geq \int_{0}^{\varepsilon R} \varrho H_*(\varrho)d\varrho$$

.

Letting $R \to \infty$,$R \to \infty$, we obtain

$$\lim_{R\to\infty} \inf \int_0^R \left(\frac{r}{R}\right)^{N-1} \int_R^{+\infty} \widetilde{H}(r,\psi)d\psi dr \geq \quad (1-\varepsilon)^{N-1} \int_0^{\infty} \varrho H_*(\varrho)d\varrho.$$

Then by the arbitrariness of $\varepsilon > 0$,$\varepsilon > 0$, we

$$\lim_{R\to\infty} \inf \int_0^R \left(\frac{r}{R}\right)^{N-1} \int_R^{+\infty} \widetilde{H}(r,\psi)d\psi dr \geq \quad \int_0^{\infty} \varrho H_*(\varrho)d\varrho.$$

Combining this with (9)(9) and (13)(13) gives

$$\lim_{R\to\infty} \inf \int_0^R \left(\frac{r}{R}\right)^{N-1} \int_R^{+\infty} \widetilde{H}(r,\psi)d\psi dr = \int_0^{\infty} \varrho H_*(\varrho)\, d\varrho.$$

The proof is now complete [5] .

Conclusion :

Finally we prove some useful facts about the kernel function HH and the associated functions HH and H_*H_*, which pave the way for further analysis of

$$\begin{cases} u_t(t,r) = d\int_0^{h(t)} \widetilde{H}(r,\psi)u(t,\psi)d\psi - du(t,\psi) + f(t,r,u), t > 0, r \in [0,h(t)), \\ u(t,h(t)) = 0, & t > 0, \\ h'(t) = \dfrac{\mu}{h^{N-1}(t)} \displaystyle\int_0^{h(t)} \int_{h(t)}^{+\infty} \widetilde{H}(r,\psi) r^{N-1} u(t,r)\, d\psi dr, & t > 0, \\ h(0)h_0, u(0,r) = u_0(r), & r \in [0,h_0]. \end{cases}$$

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